Representation of limit values for nonexpansive stochastic differential games

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Outline

1. Objective of the talk
2. Preliminaries and the Nonexpansivity Condition for SDGs
3. Hamilton-Jacobi-Bellman-Isaacs equations
4. Convergence problems for the stochastic differential games

1. Objective of the talk
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1. Objective of the talk

Motivation: For the following controlled ODE:

$$\begin{cases} dX_t^{x,u} = g(X_t^{x,u}, u_t)dt, \quad t \geq 0, \\ X_0^{x,u} = x \in \overline{\theta}, \end{cases}$$

(Abel mean) $V_\lambda(x) := \inf_{u \in \mathcal{U}} \int_0^\infty e^{-\lambda s} \psi(X_s^{x,u}, u_s) ds, \ x \in \overline{\theta}.$

- The existence of the limit of λV_λ as $\lambda \to 0$ is very interesting, e.g., strongly connected with ergodic control, homogenization...

Recently,

- Quincampoix, Renault. SIAM J. Control Optim., 2011.
- Cannarsa, Quincampoix. SIAM J. Control Optim., 2015.
- Buckdahn, Quincampoix, Renault. J. Differential Equations, 2015.

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- Li, Zhao. SPA, accepted, 2018.

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Objective of the talk

In order to introduce the studied stochastic differential games we consider

+ W–a standard d-dimensional B.M. over a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$; $\mathbb{F}$–the completed filtration generated by W;
+ The control state spaces of the Players 1 and 2: U, V–compact metric spaces;
+ $\mathcal{U} = L^0_{\mathbb{F}}(\mathbb{R}_+; U)$ (resp. $\mathcal{V} = L^0_{\mathbb{F}}(\mathbb{R}_+; V)$)–the space of the admissible controls for Player 1 (resp. Player 2);
+ $\mathcal{A}$–the set of non anticipative strategies $\alpha : \mathcal{V} \to \mathcal{U}$ for Player 1;
+ $\mathcal{B}$–the set of non anticipative strategies $\beta : \mathcal{U} \to \mathcal{V}$ for Player 2 (The precise definition will be given later).

Objective of the talk

Stochastic differential games: Given $x \in \mathbb{R}^N$, $(u,v) \in \mathcal{U} \times \mathcal{V}$, we consider

The dynamics

$$\begin{cases} dX_t^{x,u,v} = b(X_t^{x,u,v}, u_t, v_t)dt + \sigma(X_t^{x,u,v}, u_t, v_t)dW_t, \quad t \geq 0, \\ X_0^{x,u,v} = x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

and, for any $\lambda > 0$, the associated discounted

Cost functional $Y_0^{\lambda,x,u,v}$ is defined through the BSDE

$$\begin{aligned} Y_t^{\lambda,x,u,v} =& Y_T^{\lambda,x,u,v} + \int_t^T (\psi(X_s^{x,u,v}, \lambda Y_s^{\lambda,x,u,v}, Z_s^{\lambda,x,u,v}, u_s, v_s) - \lambda Y_s^{\lambda,x,u,v})ds \\ & - \int_t^T Z_s^{\lambda,x,u,v} dW_s, \ 0 \leq t \leq T < \infty. \end{aligned} \tag{1.2}$$

Note, if $\psi = \psi(x,u,v)$: $Y_0^{\lambda,x,u,v} = E\Big[\int_0^{+\infty} e^{-\lambda s}\psi(X_s^{x,u,v}, u_s, v_s)ds\Big]$.

Objective of the talk

Not imposing Isaacs' condition, we consider the

Value Functions: For $\lambda > 0$, $x \in \mathbb{R}^N$,

$$V_\lambda(x) := \inf_{\alpha \in \mathcal{A}} \sup_{v \in \mathcal{V}} Y_0^{\lambda,x,\alpha(v),v}, \; x \in \mathbb{R}^N, \qquad \text{(the lower value function)} \quad (1.3)$$

$$U_\lambda(x) := \sup_{\beta \in \mathcal{B}} \inf_{u \in \mathcal{U}} Y_0^{\lambda,x,u,\beta(u)}, \; x \in \mathbb{R}^N. \qquad \text{(the upper value function)} \quad (1.4)$$

We restrict here to the study of the limit behaviour of $\lambda V_\lambda(x)$ as $\lambda \searrow 0$ for the lower value functions.

Objective of the talk

Our objectives:

• Study of conditions which guarantee the monotone convergence of $\lambda V_\lambda(\cdot)$, uniform on compacts, as $\lambda \searrow 0$. Unlike the ergodic case, the conditions will allow the limit $W_0(x) := \lim_{\lambda\to 0} \lambda V_\lambda(x)$, $x \in \mathbb{R}^N$, to depend on the initial condition $x \in \mathbb{R}^N$.

• Characterization of the limit value $W_0(\cdot)$ as the maximal viscosity subsolution of a limit PDE; an explicit representation formula is obtained under some additional conditions; this formula is namely based on

• A uniform dynamic programming principle for W_0 involving the supremum and the infimum with respect to the time over $\mathbb{R}_+$.

1 Objective of the talk

2 Preliminaries and the Nonexpansivity Condition for SDGs

3 Hamilton-Jacobi-Bellman-Isaacs equations

4 Convergence problems for the stochastic differential games

Preliminaries

Spaces we work with:

$$S^2_{\mathbb{F}}(\mathbb{R}) := \Big\{(\phi_t)_{0\le t<\infty} \text{ real-valued continuous } \mathbb{F}\text{-adapted process: } \mathbb{E}[\sup_{t\in[0,\infty)} |\phi_t|^2] < \infty\Big\};$$

$$\mathcal{H}^2_{\mathbb{F}}(\mathbb{R}^d) := \Big\{(\phi_t)_{0\le t<\infty}\ \mathbb{R}^d\text{-valued } \mathbb{F}\text{-progr. meas. process: } \mathbb{E}[\textstyle\int_0^\infty |\phi_t|^2 dt] < \infty\Big\};$$

$$\mathcal{H}^{2,-2\lambda}_{\mathbb{F}}(0,T;\mathbb{R}^d) := \Big\{(\phi_t)_{0\le t\le T}\ \mathbb{R}^d\text{-valued } \mathbb{F}\text{-progr. meas. process: } \mathbb{E}[\textstyle\int_0^T \exp(-2\lambda t)|\phi_t|^2 dt] < \infty\Big\};$$

$$\mathcal{H}^2_{loc}(\mathbb{R}^d) := \Big\{(\phi_t)_{0\le t<\infty}\ \mathbb{R}^d\text{-valued } \mathbb{F}\text{-progr. meas. process: } \mathbb{E}[\textstyle\int_0^T |\phi_t|^2 dt] < +\infty, 0 \le T < \infty\Big\};$$

$$L^\infty_{\mathbb{F}}(0,\infty;\mathbb{R}^d) := \Big\{(\phi_t)_{0\le t<\infty}\ \mathbb{R}^d\text{-valued } \mathbb{F}\text{-adapted essentially bounded process}\Big\};$$

$$L^2(\mathcal{F}_\infty;\mathbb{R}) := \Big\{\xi \text{ real-valued } \mathcal{F}_\infty\text{-meas. r.v.: } \mathbb{E}[|\xi|^2] < \infty\Big\}.$$

Preliminaries

We begin with a recall on <u>the BSDEs with infinite time horizon</u>. For its driver $\psi : \mathbb{R}_+ \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ we suppose:

$$\begin{cases} \text{(Ai) } \forall\ (y,z) \in \mathbb{R} \times \mathbb{R}^d,\ \psi(.,y,z) = (\psi(t,y,z))_{t\geq 0} \text{ is } \mathbb{F}\text{-progr. meas.;} \\ \text{(Aii) } \forall\ (t,\omega,z) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^d,\ y \mapsto \psi(t,\omega,y,z) \text{ is continuous;} \\ \text{(Aiii) } \exists\ K_z, M \geq 0 \text{ s.t.} \\ \quad (\psi(t,y,z) - \psi(t,y',z))(y-y') \leq 0,\ dtdP\text{-a.e.}, \forall\ y,y' \in \mathbb{R},\ z \in \mathbb{R}^d; \\ \quad |\psi(t,y,z) - \psi(t,y,z')| \leq K_z|z-z'|,\ dtdP\text{-a.e.}, \forall\ y \in \mathbb{R},\ z,\ z' \in \mathbb{R}^d; \\ \quad |\psi(t,y,0)| \leq M,\ dtdP\text{-a.e.}, \forall\ y \in \mathbb{R}. \end{cases} \tag{A1}$$

Preliminaries

For any given $\lambda > 0$ we consider the BSDE on the infinite time interval:

$$Y_t^\lambda = Y_T^\lambda + \int_t^T (\psi(s, \lambda Y_s^\lambda, Z_s^\lambda) - \lambda Y_s^\lambda)ds - \int_t^T Z_s^\lambda dW_s, \ 0 \le t \le T < \infty. \quad (2.1)$$

Definition 1

A couple of processes (Y^λ, Z^λ) is called a solution of BSDE (2.1) on the infinite time interval, if $Y^\lambda = (Y_t^\lambda)_{t\ge 0} \in L_{\mathbb{F}}^\infty(0,\infty;\mathbb{R})$, $Z^\lambda = (Z_t^\lambda)_{t\ge 0} \in \mathcal{H}_{\mathbb{F},loc}^2(\mathbb{R}^d)$ (i.e., $Z^\lambda I_{[0,T]} \in \mathcal{H}_{\mathbb{F}}^2(0,T;\mathbb{R}^d)$, for all $T > 0$), and (Y^λ, Z^λ) satisfies eq. (2.1).

Proposition 1

Under our assumptions on ψ, BSDE (2.1) on the infinite time interval has a unique solution $(Y^\lambda, Z^\lambda) \in L_{\mathbb{F}}^\infty(0,\infty;\mathbb{R}) \times \mathcal{H}_{\mathbb{F},loc}^2(\mathbb{R}^d)$. Moreover, we have

$$|Y_t^\lambda| \le \frac{M}{\lambda}, \ t \ge 0, \text{ and } \mathbb{E}[\int_0^\infty |e^{-\lambda t} Z_t^\lambda|^2 dt] \le 2(\frac{M}{\lambda})^2(2 + \frac{K_z^2}{\lambda}).$$

Preliminaries

Remark

For ψ satisfying assumption (A1) and, $\forall\ n \geq 1$, define the supremum convolution

$$\psi_n(t,\omega,y,z) := \sup\{\psi(t,\omega,y',z) - n(y-y')^+,\ y' \in \mathbb{R}\},$$

$(t,\omega,y,z) \in \mathbb{R}_+ \times \Omega \times \mathbb{R} \times \mathbb{R}^d$. Then, for all $n \geq 1$, ψ_n satisfies (A1), it's Lipsch. w.r.t. y with Lipschitz constant n, and

$$\psi(t,\omega,y,z) \leq \psi_n(t,\omega,y,z) \searrow \psi(t,\omega,y,z)\ (n \to +\infty),$$

$(t,\omega,y,z) \in \mathbb{R}_+ \times \Omega \times \mathbb{R} \times \mathbb{R}^d$. This pointwise convergence is non-increasing and bounded by M.

For ψ, standard arguments reduce the proof of the preceding proposition to the case of drivers ψ_n. (Similar methods in Lepeltier, San Martin (1997)).

In particular, this allows to prove also the following comparison result for BSDEs with infinite time horizon. Then, the uniqueness is a direct consequence of the following comparison result.

Lemma 1

Let the coefficients ψ_i, $i = 1, 2$, satisfy (A1) and be such that $\psi_1 \leq \psi_2$. Then, if (Y^i, Z^i) denotes the solution of BSDE (2.1) with coefficient ψ_i, we have $Y_t^1 \leq Y_t^2$, $t \geq 0$, P-a.s.

Preliminaries

The setting of our stochastic differential games (SDGs):

We begin with its dynamics:

Let $b : \mathbb{R}^N \times U \times V \to \mathbb{R}^N$, $\sigma : \mathbb{R}^N \times U \times V \to \mathbb{R}^{N\times d}$ satisfy the assumptions:

$$\begin{cases} \text{(Hi) } b,\ \sigma \text{ are uniformly continuous on } \mathbb{R}^N \times U \times V; \\ \text{(Hii) } \exists\ c > 0 \text{ s.t., } \forall\ x,\ x' \in \mathbb{R}^N,\ u \in U,\ v \in V, \\ \quad |b(x,u,v) - b(x',u,v)| + |\sigma(x,u,v) - \sigma(x',u,v)| \leq c|x - x'|, \\ \quad |b(x,u,v)| + |\sigma(x,u,v)| \leq c. \end{cases} \tag{H1}$$

Given $x \in \mathbb{R}^N$ and $(u,v) \in \mathcal{U} \times \mathcal{V}$, let $X^{x,u,v} = (X_t^{x,u,v})_{t\geq 0}$ denote the unique $\mathbb{R}^N$-valued continuous, $\mathbb{F}$-adapted solution of (1.1)

$$\begin{cases} dX_t^{x,u,v} = b(X_t^{x,u,v}, u_t, v_t)dt + \sigma(X_t^{x,u,v}, u_t, v_t)dW_t, \quad t \geq 0, \\ X_0^{x,u,v} = x \in \mathbb{R}^N. \end{cases} \tag{2.2}$$

Remark

From SDE standard estimates we have that, for all $T > 0$, and $k \geq 2$, there exists $C_k(T) > 0$ s.t. for all $t \in [0, T]$, x, $x' \in \mathbb{R}^N, u \in \mathcal{U}, v \in \mathcal{V}$,

$$\mathbb{E}[\sup_{0 \leq s \leq T} |X_s^{x,u,v}|^k] \leq C_k(T)(1 + |x|^k);$$

$$\mathbb{E}[\sup_{0 \leq s \leq T} |X_s^{x,u,v} - X_s^{x',u,v}|^k] \leq C_k(T)|x - x'|^k.$$

(see, Ikeda, Watanabe, pp.166-168 or Karatzas, Shreve, pp.289-290).

Let us come now to the cost functional defined through a BSDE on the infinite time horizon $[0, +\infty)$.

Preliminaries

BSDE: For any $\lambda > 0$, $x \in \mathbb{R}^N$ and $(u, v) \in \mathcal{U} \times \mathcal{V}$, we consider the BSDE

$$\begin{aligned} Y_t^{\lambda,x,u,v} =& Y_T^{\lambda,x,u,v} + \int_t^T (\psi(X_s^{x,u,v}, \lambda Y_s^{\lambda,x,u,v}, Z_s^{\lambda,x,u,v}, u_s, v_s) - \lambda Y_s^{\lambda,x,u,v}) ds \\ & - \int_t^T Z_s^{\lambda,x,u,v} dW_s, \ 0 \le t \le T < \infty. \end{aligned} \tag{2.3}$$

whose coefficient $\psi : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^d \times U \times V \to \mathbb{R}$ is supposed to satisfy:

$$\begin{cases} \text{(Hiii) } \psi \text{ is uniformly continuous on } \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^d \times U \times V; \\ \text{(Hiv) } \exists\, K_x, K_z \text{ and } M \ge 0 \text{ s.t.} \\ \quad (\psi(x, y, z, u, v) - \psi(x, y', z, u, v))(y - y') \le 0, \\ \quad |\psi(x, y, z, u, v) - \psi(x', y, z', u, v)| \le K_x|x - x'| + K_z|z - z'|, \\ \quad |\psi(x, y, 0, u, v)| \le M, \quad (x, x', y, y', z, z', u, v) \in \mathbb{R}^{2N+2+2d} \times U \times V. \end{cases} \tag{H2}$$

Preliminaries

Remark: From Proposition 1 we know that there is a unique solution $(Y^{\lambda,x,u,v}, Z^{\lambda,x,u,v}) \in L^{\infty}_{\mathbb{F}}(0,\infty;\mathbb{R}) \times \mathcal{H}^2_{loc}(\mathbb{R}^d)$, and $|Y_t^{\lambda,x,u,v}| \leq \frac{M}{\lambda},\ t \geq 0$.

Cost functional: $Y_0^{\lambda,x,u,v}$, $(u,v) \in \mathcal{U} \times \mathcal{V}$.

Which type of game: • If "control against control", then, in general, even no dynamic programming principle;

• "Non-anticipative strategy against control": Fleming, Souganidis, 1989; Buckdahn,Li, 2008; "Non-anticipative strategy with delay against non-anticipative strategy with delay": Buckdahn; Cardaliaguet, Rainer,...; both concepts give the same value functions.

• "Non-anticipative strategy against control" seems to be well adapted to our studies.

Preliminaries

Definition 2 (Non-anticipative strategies)

1) A mapping $\alpha : \mathcal{V} \to \mathcal{U}$ is an admissible strategy for Player 1, if it is non-anticipating in the following sense: For all stopping time $\tau : \Omega \to \mathbb{R}_+$ and all controls $v, v' \in \mathcal{V}$ it holds that, if $v = v'$ $dsd\mathbb{P}$-a.e. on the stochastic interval $[[0, \tau]]$, then also $\alpha(v) = \alpha(v')$ $dsd\mathbb{P}$-a.e. on $[[0, \tau]]$.

The set of all admissible strategies for Player 1 is denoted by $\mathcal{A}$.

2) Symmetrically to the definition of admissible strategies for Player 1, those for Player 2 are the non-anticipating mappings $\beta : \mathcal{U} \to \mathcal{V}$.

We denote by $\mathcal{B}$ the set of all admissible strategies for Player 2.

Preliminaries

Value Function: For $\lambda > 0$,

$$V_\lambda(x) := \inf_{\alpha \in \mathcal{A}} \sup_{v \in \mathcal{V}} Y_0^{\lambda, x, \alpha(v), v}, \ x \in \mathbb{R}^N, \qquad \text{(the lower value function)} \quad (2.4)$$

$$U_\lambda(x) := \sup_{\beta \in \mathcal{B}} \inf_{u \in \mathcal{U}} Y_0^{\lambda, x, u, \beta(u)}, \ x \in \mathbb{R}^N. \qquad \text{(the upper value function)} \quad (2.5)$$

We concentrate here on the study of the limit behaviour of $\lambda V_\lambda(\cdot)$, as $\lambda \searrow 0$.

Preliminaries

Crucial in our approach are the following conditions:

Nonexpansivity condition: $\exists\ \bar{c}_0 > 0$ s.t., for all $x,\ \bar{x} \in \mathbb{R}^N, u \in U$, $\exists\ \bar{u} \in U$ s.t., for all $v \in V,\ y \in \mathbb{R},\ z \in \mathbb{R}^d$,

$$\begin{cases} \text{(i) } g(x,\bar{x},u,\bar{u},v) := \langle x - x', b(x,u,v) - b(\bar{x},\bar{u},v)\rangle + \frac{1}{2}|\sigma(x,u,v) - \sigma(\bar{x},\bar{u},v)|^2 \\ \qquad\qquad +K_z|\sigma(x,u,v) - \sigma(\bar{x},\bar{u},v)||x - \bar{x}| \leq 0; \\ \text{(ii) } \widetilde{\psi}(x,\bar{x},u,\bar{u},v,y,z) := |\psi(x,y,z,u,v) - \psi(\bar{x},y,z,\bar{u},v)| - \bar{c}_0|x - \bar{x}| \leq 0, \end{cases} \tag{H3}$$

with $K_z > 0$ introduced in assumption (H2).

Stochastic nonexpansivity condition: For all $\varepsilon > 0,\ \lambda > 0,\ x,\ \bar{x} \in \mathbb{R}^N$ and all $\alpha \in \mathcal{A}$, $\exists\ \bar{\alpha} \in \mathcal{A}$ s.t., for all $v \in \mathcal{V}$ and all $\gamma \in L^\infty_{\mathbb{F}}(0,\infty;\mathbb{R}^d)$ with $|\gamma_s| \leq K_z$, dsdP-a.e., with the notation $L^\gamma_t = \exp\{\int_0^t \gamma_s dW_s - \frac{1}{2}\int_0^t |\gamma_s|^2 ds\}$,

$$\begin{cases} \text{(i) } \left(\mathbb{E}[L^\gamma_t|X^{x,\alpha(v),v}_t - X^{\bar{x},\bar{\alpha}(v),v}_t|^2]\right)^{\frac{1}{2}} \leq |x - \bar{x}| + \varepsilon,\ t \geq 0; \\ \text{(ii) } |\lambda Y^{\lambda,x,\alpha(v),v}_0 - \lambda Y^{\lambda,\bar{x},\bar{\alpha}(v),v}_0| \leq \bar{c}_0|x - \bar{x}| + \varepsilon. \end{cases} \tag{H4}$$

Remark

1) The nonexpansivity condition is in particular satisfied under the very strong condition

$$\langle x-\bar{x}, b(x,u,v)-b(\bar{x},u,v)\rangle \leq -(C_\sigma^2/2 + C_\sigma K_z)|x-\bar{x}|^2,$$

for all $x, \bar{x} \in \mathbb{R}^N, (u,v) \in U \times V$, where C_σ is the Lipschitz constant of $x \to \sigma(x,u,v)$.

Indeed, it suffices to choose $\bar{u} = u$ and $\bar{c}_0$ as Lipschitz constant of $\psi(x,y,z,u,v)$.

2) Our nonexpansivity condition generalizes that for stochastic control problems studied by Li and Zhao (2017) (they have been the first to consider this assum. for cost functionals defined through a BSDE). Before, for classical stochastic control problems it was introduced by Buckdahn, Goreac and Quincampoix (2013). As we will show for our framework of SDGs, the cited authors have shown that the nonexpansivity condition implies the stochastic one. However, in the case of SDGs, the proof is much more involved and very technical.

Preliminaries

Theorem 1

We suppose our standard assumptions (H1) and (H2) on the coefficients b, σ and ψ. Then the nonexpansivity condition (H3) implies the stochastic nonexpansivity condition (H4).

An immediate consequence of the above theorem is the following result.

Lemma 5

We suppose that our standard assumptions (H1), (H2) and the nonexpansivity condition (H3) are satisfied. Then the family of functions $\{\lambda V_\lambda\}_{\lambda>0}$ is equicontinuous and equibounded on $\mathbb{R}^N$. More precisely, for the constants $\overline{c}_0 > 0$, $M > 0$ defined in (H2) and (H3), it holds that, for all $\lambda > 0$, and for all $x, x' \in \mathbb{R}^N$,

$$\begin{cases} \text{(i) } |\lambda V_\lambda(x) - \lambda V_\lambda(x')| \leq \overline{c}_0 |x - x'|, \\ \text{(ii) } |\lambda V_\lambda(x)| \leq M. \end{cases}$$

1. Objective of the talk
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Hamilton-Jacobi-Bellman-Isaacs equations

Recall that $\mathcal{S}^N$ is the set of symmetric real $N \times N$ matrices. Let us consider a Hamiltonian which can but needs not to be necessarily related with our SDGs.

Hamiltonian: $H : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N \to \mathbb{R}$.

Assumptions: Let H be a uniformly continuous function satisfying

(A_H) (i) (Monotonicity property) $H(x, r, p, A) \leq H(x, s, p, B)$, for all $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$, (r, A), $(s, B) \in \mathbb{R} \times \mathcal{S}^N$ with $r \leq s$ and $B \leq A$.

(ii) (Modulus of continuity) $\exists\ \rho : \mathbb{R}_+ \to \mathbb{R}_+$, $\rho(0+) = 0$, s.t.,

$|H(\bar{x}, r, \alpha(x - \bar{x}), B) - H(x, r, \alpha(x - \bar{x}), A)| \leq \rho(\alpha|x - \bar{x}|^2 + |x - \bar{x}|)$,

for all $r \in \mathbb{R}$, $x, \bar{x} \in \mathbb{R}^N$, $\alpha > 0$ and all A, $B \in \mathcal{S}^N$ s.t.

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

where I denotes the unit matrix in $\mathbb{R}^{N \times N}$.

(iii) (Boundedness) $|H(x, r, 0, 0)| \leq \bar{M}$, for all $(x, r) \in \mathbb{R}^N \times \mathbb{R}$, for some constant $\bar{M} \in \mathbb{R}_+$.

Hamilton-Jacobi-Bellman-Isaacs equations

For $\lambda > 0$ we consider the PDE

$$\lambda V(x) + H(x, \lambda V(x), DV(x), D^2V(x)) = 0,\ \ x \in \mathbb{R}^N. \tag{3.1}$$

Remark

• Under the above assumptions (A_H) (i)-(ii) the above PDE obeys the comparison principle: The uniformly continuous viscosity subsolutions of (3.1) are less than or equal to the uniform continuous viscosity supersolutions.

• (A_H) (iii) ensures that $U(x) = -\bar{M}/\lambda$, $x \in \mathbb{R}^N$, is a viscosity supersolution. Combined with the comparison principle, this allows to use Perron's method: PDE (3.1) has a unique viscosity solution given by

$$V(x) = \sup\left\{U(x) : U \text{ is a subsolution of (3.1) and } |U| \le \bar{M}/\lambda\right\},\ x \in \mathbb{R}^N.$$

Hamilton-Jacobi-Bellman-Isaacs equations

Let us introduce the space

$$Lip_{M_0}(\mathbb{R}^N) := \big\{U : \mathbb{R}^N \to \mathbb{R} \big| |U(x)| \leq M_0, \\ |U(x) - U(\bar{x})| \leq M_0 |x - \bar{x}|, x, \bar{x} \in \mathbb{R}^N \big\}$$

and let us suppose that, for $M_0 > 0$ large enough,

(H) For the unique viscosity solution V_λ of PDE (3.1), λV_λ belongs to $Lip_{M_0}(\mathbb{R}^N)$, for all $\lambda > 0$.

Remark

Under the assumptions (A_H) (i)-(iii) PDE (3.1) has a unique continuous viscosity solution V_λ satisfying $|\lambda V_\lambda(x)| \leq \bar{M}$, for all $x \in \mathbb{R}^N, \ \lambda > 0.$ From Lemma 5, we see that, when the Hamiltonian is associated with a SDG and the non-expansivity condition (H3) is satisfied, assumption (H) holds true.

Hamilton-Jacobi-Bellman-Isaacs equations

In addition to (H) we also need the Radial Monotonicity Condition (RM) for the Hamiltonian H:

(RM) $H(x, r, \ell p, \ell A) \geq H(x, r, p, A)$, $\ell \geq 1$, $(x, r, p, A) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N$.

Theorem 2

Let the Hamiltonian H satisfy the assumptions $\mathbf{(}A_H\mathbf{)}$, (H) and also (RM). Then

(i) $\lambda \to \lambda V_\lambda(x)$ *is nondecreasing, for all* $x \in \mathbb{R}^N$*;*

(ii) *The limit* $W_0(x) := \lim_{\lambda \to 0^+} \lambda V_\lambda(x)$ *exists, for all* $x \in \mathbb{R}^N$*;*

(iii) *The convergence in* (ii) *is uniform on compacts in* $\mathbb{R}^N$*.*

Hamilton-Jacobi-Bellman-Isaacs equations

Proof: Let us introduce the family of Hamiltonians

$$H_\lambda(x,r,p,A) := \lambda H(x,r,\frac{1}{\lambda}p,\frac{1}{\lambda}A),\ (x,r,p,A)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}^N,\ \lambda>0.$$

Then $\widehat{U}_\lambda(x) := \lambda V_\lambda(x)$ is, obviously, a viscosity solution of

$$\lambda\widehat{U}_\lambda(x)+H_\lambda(x,\widehat{U}_\lambda(x),D\widehat{U}_\lambda(x),D^2\widehat{U}_\lambda(x))=0. \tag{3.2}$$

On the other hand, for any $\lambda,\ \mu>0$, we have

$$\frac{\lambda}{\mu}H_\mu(x,r,\frac{\mu}{\lambda}p,\frac{\mu}{\lambda}A)=\frac{\lambda}{\mu}(\mu H(x,r,\frac{\mu}{\lambda}(\frac{1}{\mu}p),\frac{\mu}{\lambda}(\frac{1}{\mu}A)))=H_\lambda(x,r,p,A).$$

Hamilton-Jacobi-Bellman-Isaacs equations

Hence, from the radial monotonicity condition (RM) we get in viscosity sense, for all $\mu > \lambda > 0$,

$$\begin{aligned}
&\lambda\widehat{U}_\mu(x) + H_\lambda(x, \widehat{U}_\mu(x), D\widehat{U}_\mu(x), D^2\widehat{U}_\mu(x)) \\
&= \mu \cdot \frac{\lambda}{\mu}\widehat{U}_\mu(x) + \frac{\lambda}{\mu}H_\mu(x, \widehat{U}_\mu(x), \frac{\mu}{\lambda}D\widehat{U}_\mu(x), \frac{\mu}{\lambda}D^2\widehat{U}_\mu(x)) \\
&= \frac{\lambda}{\mu}[\mu\widehat{U}_\mu(x) + \mu H(x, \widehat{U}_\mu(x), \frac{\mu}{\lambda}(\frac{1}{\mu}D\widehat{U}_\mu(x)), \frac{\mu}{\lambda}(\frac{1}{\mu}D^2\widehat{U}_\mu(x)))] \\
&\geq \frac{\lambda}{\mu}[\mu\widehat{U}_\mu(x) + \mu H(x, \widehat{U}_\mu(x), \frac{1}{\mu}D\widehat{U}_\mu(x), \frac{1}{\mu}D^2\widehat{U}_\mu(x))] \\
&= \frac{\lambda}{\mu}(\mu\widehat{U}_\mu(x) + H_\mu(x, \widehat{U}_\mu(x), D\widehat{U}_\mu(x), D^2\widehat{U}_\mu(x))) = 0, \ x \in \mathbb{R}^N.
\end{aligned}$$

This shows that $\widehat{U}_\mu \in Lip_{M_0}(\mathbb{R}^N)$ is a viscosity supersolution of (3.2). From the comparison principle it follows that $\widehat{U}_\mu \geq \widehat{U}_\lambda$ on $\mathbb{R}^N$. This proves (i). Statement (ii) follows from (i) and the boundedness of λV_λ, while thanks to the fact that $\lambda V_\lambda \in \mathrm{Lip}_{M_0}(\mathbb{R}^N)$, $\lambda > 0$, the Arzelà-Ascoli Theorem yields (iii). $\square$

Hamilton-Jacobi-Bellman-Isaacs equations

When is this radial monotonicity condition satisfied? Recall Lemma 3.1, Li, Zhao (2018, SPA):

Lemma 6

Let $H(x, r, p, A)$ be convex in $(p, A) \in \mathbb{R}^N \times \mathcal{S}^N$. Then the following conditions are equivalent:

i) *The radial monotonicity condition (RM) is satisfied by* $H(x, r, \cdot, \cdot)$;

ii) $H(x, r, \ell' p, \ell' A) \geq H(x, r, \ell p, \ell A),\ 0 \leq \ell \leq \ell',\ (p, A) \in \mathbb{R}^N \times \mathcal{S}^N$;

iii) $H(x, r, p, A) \geq H(x, r, 0, 0),\ (p, A) \in \mathbb{R}^N \times \mathcal{S}^N$.

Hamilton-Jacobi-Bellman-Isaacs equations

However, the Hamiltonian $H(x, r, p, A)$ associated with SDGs is, in general not convex in (p, A). But we have:

Corollary 1

Given any index set Γ and convex Hamiltonians $H_\gamma(x, r, \cdot, \cdot) : \mathbb{R}^N \times \mathcal{S}^N \to \mathbb{R}$, $\gamma \in \Gamma$ with

$$H_\gamma(x, r, p, A) \geq H_\gamma(x, r, 0, 0), \text{ for all } (x, r, p, A) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N,$$

the Hamiltonian H defined by

$$H(x, r, p, A) = \inf_{\gamma \in \Gamma} H_\gamma(x, r, p, A), \ (x, r, p, A) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N,$$

satisfies the radial monotonicity condition (RM).

Hamilton-Jacobi-Bellman-Isaacs equations

Remark

In the case of dimension $N = 1$, as we have a standard assumption on our Hamiltonian $H = H(x, r, p, A) : \mathbb{R}^4 \to \mathbb{R}$ that it is continuous and proper, the mapping $\mathbb{R} \ni A \to H(x, r, p, A)$ has to be non-increasing. Consequently, for the radial monotonicity of $H(x, r, \cdot, \cdot)$ it is sufficient that the Hamiltonian satisfies the following conditions in addition to the standard ones mentioned above:

(i) $H(x, r, p, A) = H(x, r, p, 0)$, for all $p, A \in \mathbb{R}$ with $A \geq 0$,

(ii) $H(x, r, p, A)$ is non-decreasing in $p \in \mathbb{R}_+$ and non-increasing in $p \in \mathbb{R}_-$, for all A.

Of course, these conditions (i) and (ii) are only sufficient but not necessary for (RM). Indeed, as one checks easily, for the continuous proper Hamiltonian

$$H(x, r, p, A) := (A^- - p^+)^+, \ (p, A) \in \mathbb{R} \times \mathbb{R},$$

the condition (RM) holds true but not the above condition (ii).

Theorem 2 has established the uniform convergence on compacts and the monotone convergence of $\lambda V_\lambda(\cdot)$, as $\lambda \searrow 0$.

Theorem 3

Let us make the same assumptions as in Theorem 2. For every $\lambda > 0$ let V_λ denote the unique viscosity solution of the PDE

$$\lambda V(x) + H(x, \lambda V(x), DV(x), D^2V(x)) = 0, \;\; x \in \mathbb{R}^N, \tag{3.3}$$

such that $\lambda V_\lambda \in \text{Lip}_{M_0}(\mathbb{R}^N)$, for some $M_0 > 0$ large enough and independent of λ. Then, $W_0(x) := \lim_{\lambda \to 0^+} \lambda V_\lambda(x)$, $x \in \mathbb{R}^N$, satisfies

$$\begin{aligned} W_0(x) = \sup\{&W(x) : W \in \text{Lip}_{M_0}(\mathbb{R}^N), W + \overline{H}(x, W, DW, D^2W) \le 0 \\ &\text{on } \mathbb{R}^N \text{(in viscosity sense)}\}, \; x \in \mathbb{R}^N, \end{aligned}$$

where

$$\overline{H}(x, r, p, A) := \min\left\{M_0, \sup_{\ell > 0} H(x, r, \ell p, \ell A)\right\}.$$

Hamilton-Jacobi-Bellman-Isaacs equations

The following result gives a sufficient condition for a constant limit $W_0(\cdot)$ of $\lambda V_\lambda(\cdot)$, like in the ergodic case.

Corollary 2

In addition to the assumptions in Theorem 3 we suppose that, for all $(x, p, A) \in \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \times \mathcal{S}^N$, $\sup_{\ell > 0} H(x, W_0(x), \ell p, \ell A) = +\infty$. Then, the function $W_0(\cdot)$ must be constant on $\mathbb{R}^N$.

1. Objective of the talk
2. Preliminaries and the Nonexpansivity Condition for SDGs
3. Hamilton-Jacobi-Bellman-Isaacs equations
4. Convergence problems for the stochastic differential games

Convergence problems for the SDGs

We consider now the Hamiltonian of our SDGs introduced in the first part.

Hamiltonian: For $(x, y, p, A) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N$ we put

$$H(x,y,p,A) := \inf_{v \in V} \sup_{u \in U} \Big\{ \langle -p, b(x,u,v) \rangle - \frac{1}{2} tr(\sigma\sigma^*(x,u,v)A) - \psi(x,y,p\sigma(x,u,v),u,v) \Big\}. \quad (4.1)$$

Theorem 4

Under the assumptions (H1), (H2) and the nonexpansivity condition (H3) the lower value function $V_\lambda(x) = \inf_{\alpha \in \mathcal{A}} \sup_{v \in \mathcal{V}} Y_0^{\lambda,x,\alpha(v),v}$, $x \in \mathbb{R}^N$, defined by (2.4) of the SDG (2.2)-(2.3) is a viscosity solution of the HJBI equation

$$\lambda V(x) + H(x, \lambda V(x), DV(x), D^2V(x)) = 0, \; x \in \mathbb{R}^N, \quad (4.2)$$

where $H(x,y,p,A)$ is defined by (4.1). Moreover, the solution is unique in the class of the uniformly continuous functions on $\mathbb{R}^N$.

Convergence problems for the SDGs

Our objective is now to study for our SDG the limit behaviour of $\lambda V_\lambda(\cdot)$, as $\lambda \searrow 0$. For this end, we begin with the Dynamic Programming Principle (DPP) for $V_\lambda(\cdot)$. The DDP uses the notion of Backward stochastic semigroup(Peng (1997)): Let $\psi : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times U \times V \to \mathbb{R}$ satisfy the assumption (H2). Then, given $\lambda > 0$, $(x, u, v) \in \mathbb{R}^N \times \mathcal{U} \times \mathcal{V}$ we define for any finite time horizon $t \geq 0$ the backward stochastic semigroup

$$G_{s,t}^{\lambda,t,x,u,v}[\eta] := Y_s^{\lambda,t,\eta},\ s \in [0,t],\ \eta \in L^2(\mathcal{F}_t, \mathbb{R}),$$

where $(Y^{\lambda,t,\eta}, Z^{\lambda,t,\eta}) \in \mathcal{S}^2_{\mathbb{F}}(0,t) \times \mathcal{H}^2_{\mathbb{F}}(0,t;\mathbb{R}^d)$ is the unique solution of the BSDE

$$\left\{\begin{array}{rcl} dY_s^{\lambda,t,\eta} & = & -\big(\psi(X_s^{x,u,v}, \lambda Y_s^{\lambda,t,\eta}, Z_s^{\lambda,t,\eta}, u_s, v_s) - \lambda Y_s^{\lambda,t,\eta}\big)ds + Z_s^{\lambda,t,\eta} dW_s, \\ & & \qquad\qquad s \in [0,t], \\ Y_t^{\lambda,t,\eta} & = & \eta. \end{array}\right. \tag{4.3}$$

Convergence problems for the SDGs

Proposition 2

Under our standard assumptions (H1), (H2) and (H3) the lower value function V_λ defined by (2.4) satisfies the following DPP: For all $t \geq 0,\ x \in \mathbb{R}^N$ and all $\lambda > 0$,

$$V_\lambda(x) = \inf_{\alpha \in \mathcal{A}} \sup_{v \in \mathcal{V}} G_{0,t}^{\lambda,t,x,\alpha(v),v}\left[V_\lambda(X_t^{x,\alpha(v),v})\right]. \quad (4.4)$$

Proof: $V_\lambda(t,x) := V_\lambda(x)$, $(t,x) \in [0,T] \times \mathbb{R}^N$, $(T > 0)$, is the unique viscosity solution of the HJBI equation on $[0,T] \times \mathbb{R}^N$

$$\begin{aligned} -\partial_t V_\lambda(t,x) + \lambda V_\lambda(t,x) + H(x, \lambda V_\lambda(t,x), DV_\lambda(t,x), D^2 V_\lambda(t,x)) &= 0, \\ V_\lambda(T,x) &= V_\lambda(x). \end{aligned}$$

This kind of HJBI equation was studied by Buckdahn, Li (2008). They proved in particular the DPP: For all $t \in [0,T]$, $x \in \mathbb{R}^N$,

$$V_\lambda(0,x) = \inf_{\alpha \in \mathcal{A}} \sup_{v \in \mathcal{V}} G_{0,t}^{\lambda,t,x,\alpha(v),v}\left[V_\lambda(t, X_t^{x,\alpha(v),v})\right].$$

Recalling the definition of $V_\lambda(t,x)$, this is just what we have had to prove. $\square$

Convergence problems for the SDGs

Let us take now the

Additional assumption adding to (H2): (H2')

$$\begin{cases} \text{(Hv) } \psi(x,y,z,u,v) = \psi(x,z,u) \text{ does not depend on } y \text{ nor on } v; \\ \text{(Hvi) } \exists\ \psi_0 : \mathbb{R}^N \times U \to \mathbb{R} \text{ s.t. } \lambda\big(\psi(x, \frac{1}{\lambda}z, u) - \psi(x,0,u)\big) \to \psi_0(z,u), \\ \quad \text{uniformly on compacts in } \mathbb{R}^N \times \mathbb{R}^d \times U. \end{cases}$$

Remark

- From the assumptions (H2) on ψ it follows that ψ_0 is independent of x.
- Moreover,

(i) $|\psi_0(z,u)| \leq K_z|z|$, $(z,u) \in \mathbb{R}^N \times U$, and

(ii) $\psi_0(az,u) = a\psi_0(z,u)$, for all $a \geq 0$ and $(z,u) \in \mathbb{R}^N \times U$.

Convergence problems for the SDGs

The associated backward stochastic semigroup: Given $x \in \mathbb{R}^N$ and $u \in \mathcal{U}$, we define $G^{t,u}_{s,t}[\eta] := \bar{Y}^{t,u}_s(\eta)$, $0 \le s \le t$, $\eta \in L^2(\mathcal{F}_t;\mathbb{R})$, through the BSDE

$$\begin{cases} d\bar{Y}^{t,u}_s(\eta) = -\psi_0(\bar{Z}^{t,u}_s(\eta), u_s)ds + \bar{Z}^{t,u}_s(\eta)dW_s, \ s \in [0,t], \\ \bar{Y}^{t,u}_t(\eta) = \eta. \end{cases} \tag{4.5}$$

Convergence problem for the SDG

As by now well known, $G^u_{s,t}[\cdot] := G^{t,u}_{s,t}[\cdot]$ satisfies the following properties of a conditional g-expectation (see Peng, 1997):

Lemma 7

Under the assumptions (H2) and (H2')

i) $G^{t,u}_{s,t}[\eta+\theta] = G^{r,u}_{s,r}[\eta]+\theta$, $0 \le s \le r \le t,\ \eta \in L^2(\mathcal{F}_r;\mathbb{R}),\ \theta \in L^2(\mathcal{F}_s;\mathbb{R})$.

ii) $G^{r,u}_{s,r}\left[G^{t,u}_{r,t}[\eta]\right] = G^{t,u}_{s,t}[\eta]$, $0 \le s \le r \le t,\ \eta \in L^2(\mathcal{F}_t;\mathbb{R})$.

iii) $G^{t,u}_{s,t}[\eta_1] \le G^{t,u}_{s,t}[\eta_2]$, for all $0 \le s \le t,\ \eta_1,\ \eta_2 \in L^2(\mathcal{F}_t;\mathbb{R})$ with $\eta_1 \le \eta_2$.

Moreover,

iv) If $z \to \psi(x,z,u)$ is concave, $G^{t,u}_{s,t}[\cdot]$ is concave over $L^2(\mathcal{F}_r;\mathbb{R})$.

For $s=0$ we have the g-expectation $G^u[\eta] := G^{t,u}_{0,t}[\eta]$, $\eta \in L^2(\mathcal{F}_t;\mathbb{R})$.

Convergence problem for the SDG

Recall that we assume $\psi(x,y,z,u,v)=\psi(x,z,u)$ does not depend on (y,v).

Theorem 5

We suppose that the assumptions (H1), (H2), (H2'), (H3) and (RM) hold true.

Then $W_0(x)=\lim_{\lambda\searrow 0}\lambda V_\lambda(x),\ x\in\mathbb{R}^N$, satisfies the DPP

$$W_0(x)=\inf_{\alpha\in\mathcal{A}}\sup_{v\in\mathcal{V}} G^{\alpha(v)}\left[W_0(X_t^{x,\alpha(v),v})\right],\ x\in\mathbb{R}^N,\ t>0. \tag{4.6}$$

Moreover, if $z\to\psi(x,z,u)$ is concave for all (x,u) and

$$\lambda\Big(\psi(x,\frac{1}{\lambda}z,u)-\psi(x,0,u)\Big)\geq\psi_0(z,u),\ \text{for all } \lambda>0,(x,u,z), \tag{H.2''}$$

then $W_0(\cdot)$ has the following representation formula:

$$W_0(x)=\inf_{\alpha\in\mathcal{A}}\sup_{v\in\mathcal{V}}\inf_{t\geq 0} G^{\alpha(v)}\left[\bar\psi(X_t^{x,\alpha(v),v})\right],\ x\in\mathbb{R}^N. \tag{4.7}$$

Here $\bar\psi(x)=\min_{u\in U}\psi(x,0,u)$.

For control problems (4.7) was studied by Li, Zhao:

$W_0(x)=\inf_{u\in\mathcal{U}}\inf_{t\geq 0}G^{\alpha(v)}\left[\bar\psi(X_t^{x,u})\right]$; difficulty for us: iteration of inf-sup-inf.

Convergence problems for the SDGs

Example for (H.2'')

Let $g : R^m \to R$ be Lipschitz, positive homogeneous, concave and superadditive (i.e., $g(a+b) - g(a) \geq g(b)$, for all $a, b \in \mathbb{R}^m$; an easy example for such $g:\ g(a) = -|a|,\ a \in \mathbb{R}^m$), and let, for suitable functions $\psi_i(x,u),\ i = 1, 2$, and $\psi_3(u)$:

$$\psi(x,z,u) := \psi_1(x,u) + g\big(\psi_2(x,u) + \psi_3(u)z\big).$$

Then, $\lambda\big(\psi(x, \frac{1}{\lambda}z, u) - \psi(x,0,u)\big) = g\big(\lambda\psi_2(x,u) + \psi_3(u)z\big) - g\big(\lambda\psi_2(x,u)\big)$, and, thus,

- $\lambda\big(\psi(x, \frac{1}{\lambda}z, u) - \psi(x,0,u)\big) \to \psi_0(z,u) := g\big(\psi_3(u)z\big)$, as $\lambda \searrow 0$, and
- $\lambda\big(\psi(x, \frac{1}{\lambda}z, u) - \psi(x,0,u)\big) \geq \psi_0(z,u) := g\big(\psi_3(u)z\big)$.

Proof of DPP (4.6) in Theorem 5: by passing to the limit in the BSDE associated with the backward stochastic semigroup used for the DPP for $V_\lambda(\cdot)$.

Convergence problems for the SDGs

The proof of (4.7) is crucially based on

Theorem 6

Let us assume that the assumptions (H1), (H2), (H2'), (H3) and (RM) are satisfied. Then we have the following strong version of the DPP: for all $x \in \mathbb{R}^N$,

$$W_0(x) = \inf_{\alpha\in\mathcal{A}} \sup_{v\in\mathcal{V}} \inf_{t\geq 0} G^{\alpha(v)}\left[W_0(X_t^{x,\alpha(v),v})\right] = \inf_{\alpha\in\mathcal{A}} \sup_{v\in\mathcal{V}} \sup_{t\geq 0} G^{\alpha(v)}\left[W_0(X_t^{x,\alpha(v),v})\right]. \tag{4.8}$$

The proof of the theorem is rather technical and long; it uses and extends the techniques developed for the proof of the DPP for SDGs (but without $\inf_{t\geq 0}$, $\sup_{t\geq 0}$) by Buckdahn, Li (2008).

Sketch of the proof of (4.7) of Theorem 5:

$$W_0(x) = \inf_{\alpha\in\mathcal{A}} \sup_{v\in\mathcal{V}} \inf_{t\geq 0} G^{\alpha(v)}\left[\bar\psi(X_t^{x,\alpha(v),v})\right],\ x \in \mathbb{R}^N,$$

with $\bar\psi(x) = \min_{u\in U} \psi(x,0,u)$.

Convergence problems for the SDGs

Step 1. To prove: $$W_0(x) \le \inf_{\alpha\in\mathcal{A}} \sup_{v\in\mathcal{V}} \inf_{t\ge 0} G^{\alpha(v)}\big[\bar\psi(X_t^{x,\alpha(v),v})\big],\ x\in\mathbb{R}^N. \quad (4.9)$$

From Theorem 3, in viscosity sense,

$$W_0(x) + \overline{H}(x, W_0(x), DW_0(x), D^2W_0(x)) \le 0,\ \ x\in\mathbb{R}^N.$$

Hence, if $J^{2,+}W_0(x)\ne\emptyset$, from the RM condition, for any $(p,A)\in J^{2,+}W_0(x)$,

$$\begin{aligned} 0 \ge& W_0(x) + \overline{H}(x, W_0(x), p, A) \ge W_0(x) + \overline{H}(x, W_0(x), 0, 0) \\ =& W_0(x) + \sup_{u\in U}(-\psi(x,0,u)), \quad \text{i.e.,} \end{aligned}$$

$$W_0(x) \le \bar\psi(x) := \min_{u\in U}\psi(x,0,u),\ \text{ if } J^{2,+}W_0(x)\ne\emptyset. \quad (4.10)$$

It can be shown that $\{x\in\mathbb{R}^N:\ J^{2,+}W_0(x)\ne\emptyset\}$ is dense in $\mathbb{R}^N$, i.e., (4.10) holds for all $x\in\mathbb{R}^N$. Thus, from (4.8),

$$\begin{aligned} W_0(x) &= \inf_{\alpha\in\mathcal{A}}\sup_{v\in\mathcal{V}}\inf_{t\ge0} G^{\alpha(v)}\big[W_0(X_t^{x,\alpha(v),v})\big] \\ &\le \inf_{\alpha\in\mathcal{A}}\sup_{v\in\mathcal{V}}\inf_{t\ge0} G^{\alpha(v)}\big[\bar\psi(X_t^{x,\alpha(v),v})\big]. \end{aligned}$$

Convergence problem for the SDG

Step 2. We show

$$W_0(x) \geq \inf_{\alpha\in\mathcal{A}} \sup_{v\in\mathcal{V}} \inf_{t\geq 0} G^{\alpha(v)}\big[\bar{\psi}(X_t^{x,\alpha(v),v})\big],\ x\in\mathbb{R}^N. \tag{4.11}$$

Recall that $V_\lambda(x)=\inf_{\alpha\in\mathcal{A}}\sup_{v\in\mathcal{V}} Y_0^{\lambda,u,v}$ is defined through our BSDE on $[0,+\infty)$ with solution $(Y^{\lambda,u,v},Z^{\lambda,u,v})$. Then $(\bar{Y}^{\lambda,x,u,v},\bar{Z}^{\lambda,x,u,v})=\lambda(Y^{\lambda,u,v},Z^{\lambda,u,v})$ is the unique solution of the BSDE on $[0,+\infty)$

$$d\bar{Y}_s^{\lambda,x,u,v} = -\lambda\Big(\psi(X_s^{x,u,v},\frac{1}{\lambda}\bar{Z}_s^{\lambda,x,u,v},u_s)-\bar{Y}_s^{\lambda,x,u,v}\Big)ds+\bar{Z}_s^{\lambda,x,u,v}dW_s,\ \ s\geq 0. \tag{4.12}$$

On the other hand,

$$\begin{aligned}\lambda\psi(x,\frac{1}{\lambda}z,u) &\geq \lambda\bar{\psi}(x)+\lambda\Big(\psi(x,\frac{1}{\lambda}z,u)-\psi(x,0,u)\Big)\\ &\geq \lambda\bar{\psi}(x)+\psi_0(z,u),\ \text{for all } (x,z,u).\end{aligned}$$

Convergence problem for the SDG

Thus, comparing the above BSDE on $[0,+\infty)$ with

$$d\widetilde{Y}_s^{\lambda,x,u,v} = -\big(\lambda\big(\bar{\psi}(X_s^{x,u,v}) - \widetilde{Y}_s^{\lambda,x,u,v}\big) + \psi_0(\widetilde{Z}_s^{\lambda,x,u,v}, u_s)\big)ds + \widetilde{Z}_s^{\lambda,x,u,v}dW_s,\ s \geq 0,$$

we see $\bar{Y}_t^{\lambda,x,u,v} \geq \widetilde{Y}_t^{\lambda,x,u,v}$, $t \geq 0$, P-a.s., for all $\lambda > 0$.

But, using the positive homogeneity of $\psi_0(\cdot,u)$, for all $t \geq 0$,

$$\widetilde{Y}_0^{\lambda,x,u,v} = \big(e^{-\lambda t}\widetilde{Y}_t^{\lambda,x,u,v} + \lambda\int_0^t e^{-\lambda s}\bar{\psi}(X_s^{x,u,v})ds\big) + \int_0^t \psi_0(e^{-\lambda s}\widetilde{Z}_s^{\lambda,x,u,v}, u_s)ds - \int_0^t e^{-\lambda s}\widetilde{Z}_s^{\lambda,x,u,v}dW_s.$$

Hence, the definition of $G^u[\cdot]$, from Lemma 7 and since $|\widetilde{Y}_t^{\lambda,x,u,v}| \leq M$ (Indeed, $|\bar{\psi}| \leq M$),

$$\begin{aligned}
\widetilde{Y}_0^{\lambda,x,u,v} &= G^u\big[e^{-\lambda t}\widetilde{Y}_t^{\lambda,x,u,v} + \lambda\int_0^t e^{-\lambda s}\bar{\psi}(X_s^{x,u,v})ds\big] \\
&\geq -e^{-\lambda t}M + \lambda\int_0^t e^{-\lambda s}G^u\big[\bar{\psi}(X_s^{x,u,v})\big]ds \\
&\to \lambda\int_0^{+\infty} e^{-\lambda s}G^u\big[\bar{\psi}(X_s^{x,u,v})\big]ds, \text{ as } t \to +\infty.
\end{aligned}$$

Convergence problem for the SDG

But this implies

$$\bar{Y}_0^{\lambda,x,u,v} \geq \widetilde{Y}_0^{\lambda,x,u,v} \geq \lambda \int_0^{+\infty} e^{-\lambda s} G^u \left[\bar{\psi}(X_s^{x,u,v})\right] ds,$$

for all $\lambda > 0$ and all $(u,v) \in \mathcal{U} \times \mathcal{V}$. Thus, as $\lambda V_\lambda(x) \to W_0(x)$ $(\lambda \searrow 0)$,

$$\begin{aligned} \big(W_0(x) \quad \leftarrow \big) \quad \lambda V_\lambda(x) &= \inf_{\alpha\in\mathcal{A}} \sup_{v\in\mathcal{V}} \bar{Y}_0^{\lambda,x,\alpha(v),v} \\ &\geq \inf_{\alpha\in\mathcal{A}} \sup_{v\in\mathcal{V}} \lambda \int_0^{+\infty} e^{-\lambda s} G^{\alpha(v)} \left[\bar{\psi}(X_s^{x,\alpha(v),v})\right] ds \\ &\geq \inf_{\alpha\in\mathcal{A}} \sup_{v\in\mathcal{V}} \inf_{s\geq 0} G^{\alpha(v)} \left[\bar{\psi}(X_s^{x,\alpha(v),v})\right], \end{aligned}$$

for all $\lambda > 0$. This proves the stated inequality.

Step 3. Combining the results of the Steps 1 and 2 we obtain

$$W_0(x) = \inf_{\alpha\in\mathcal{A}} \sup_{v\in\mathcal{V}} \inf_{t\geq 0} G^{\alpha(v)} \left[\bar{\psi}(X_t^{x,\alpha(v),v})\right], \; x \in \mathbb{R}^N.$$

The proof is complete.

Thank you very much!

谢谢!