On mean field games with a major player

P. Cardaliaguet

Paris-Dauphine

Based on ongoing works with M. Cirant (Padova) and A. Porretta (Roma Tor Vergata)

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Mean Field Games (MFG) are Nash equilibria in

- nonatomic games = infinitely many agents having individually a negligible influence on the global system (as in Schmeidler (1973), or Mas-Colell (1983, 1984))
- in a optimal control framework = each agent acts on his state which evolves in continuous time and has a payoff depending on the other's position (stochastic optimal control)

Pioneering works :

— Models invented by Lasry-Lions (2006) and Caines-Huang-Malhamé (2006)

— Similar models in the economic literature : heterogeneous agent models (Aiyagari ('94), Bewley ('86), Krusell-Smith ('98),...)

MFG with major players

In this talk we consider

- a crowd of small agents
- interacting with one or several major agents.

Motivation : models problems in which the agents have different size.

Large literature on the subject :

- Pioneering work : Huang (2010), Caines and al. (2013,...),
- Buckdahn-Li-Peng : optimal control (2014)
- Bensoussan-Chau-Yam : Stackelberg games (2015, 2016)
- Carmona and al. : probabilistic approach (2016, 2017)
- Lasry-Lions : PDE approach (2018)
- ...

Aim of the talk

Issues :

- Different approaches $\longrightarrow$ different notions of equilibria ?
- Limit of the $N-$player problem as $N \to +\infty$ $\longrightarrow$ yet another notions ?

Goal of the talk :

- Understand the relation between the different definitions of equilibria
- Investigate the limit of the $N-$player games as $N \to +\infty$

$\longrightarrow$ setting : MFG with one major player

Outline

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The game

Notation :

- $x \in \mathbb{R}^d$ is the position of a typical small player, $x_0 \in \mathbb{R}^{d_0}$ is the position of the major player,
- $m \in \mathcal{P}_2(\mathbb{R}^d)$ is the distribution of the small players, μ_0 the initial distribution.

(Feedback) strategies :

- Strategies of the small players : $\alpha = \alpha_t(x, x_0, m)$,
- Strategies of the major player : $\alpha^0 = \alpha^0_t(x_0, m)$.

Goal of the players : to **minimize**

- for the small player :

$$J(\alpha; X^0_t, (m_t)) = \mathbf{E}\left[\int_0^T L(X_t, X^0_t, \alpha_t(t, X_t, X^0_t, m_t), m_t)dt + G(X_T, X^0_T, m_T)\right],$$

where $dX_t = \alpha_t(t, X_t, X^0_t, m_t)dt + \sqrt{2}dB_t$, m_t being the distribution of the players,

- for the major player :

$$J^0(\alpha^0; (m_t)) = \mathbf{E}\left[\int_0^T L^0(X^0_t, \alpha^0_t(t, X^0_t, m_t), m_t)dt + G^0(X^0_T, m_T)\right],$$

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  where $dX^0_t = \alpha^0_t(X^0_t, m_t)dt + \sqrt{2}dB^0_t$.

Approach 1 : Nash Equilibria (Carmona and Wang)

Definition. Given an initial measure $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and an initial position $x_0^0 \in \mathbb{R}^{d_0}$ for the major player, a Nash equilibrium is a pair $(\bar\alpha^0, \bar\alpha)$ of strategies for the minor and major player such that :

1. $\bar\alpha$ **is optimal for each minor player :** for any α,

$$J(\bar\alpha; \bar X_t^0, \bar m_t) \leq J(\alpha; \bar X_t^0, \bar m_t),$$

where $(\bar X_t^0, \bar m_t)$ solves

$$\begin{cases} d\bar X_t^0 = \bar\alpha_t^0(\bar X_t^0, \bar m_t)dt + \sqrt{2}dB_t^0, \\ d_t \bar m_t = \left\{\Delta \bar m_t + \mathrm{div}(\bar m_t \bar\alpha_t(x, \bar X_t^0, \bar m_t))\right\} dt, \\ \bar m_0 = \mu_0, \;\; X_0^0 = x_0^0. \end{cases}$$

2. $\bar\alpha^0$ **is optimal for the major player :** for any α^0,

$$J^0(\bar\alpha^0; (\bar m_t)) \leq J^0(\alpha^0; (m_t)),$$

where (X_t^0, m_t) solves

$$\begin{cases} dX_t^0 = \alpha_t^0(X_t^0, m_t)dt + \sqrt{2}dB_t^0 \\ d_t m_t = \left\{\Delta m_t + \mathrm{div}(m_t \bar\alpha_t(x, X_t^0, m_t))\right\} dt, \\ m_0 = \mu_0, \;\; X_0^0 = x_0^0, \end{cases}$$

Main result of Carmona and al.

- Carmona-Zhu (2016)
  - First definition by Pontryagin stochastic maximum principle,
  - Approximate Nash equilibria for large finite player games,
  - Comparison with Huang's approach
- Carmona-Wang (2016-2017)
  - Definition of Nash equilibria,
  - existence/uniqueness for LQ or in short time and discrete setting
  - Numerical schemes

Approach 2 : PDE (Lasry-Lions)

A pair $(U^0, U) = (U^0(t, x_0, m), U(t, x, x_0, m))$ is the solution of the master equations for MFG with a major agent if :

$$
(\mathbf{M}) \begin{cases}
(i) & -\partial_t U^0 - \Delta_{x_0} U^0 + H^0(x_0, D_{x_0} U^0, m) - \int_{\mathbb{R}^d} \mathrm{div}_y D_m U^0(t, x_0, m, y) dm(y) \\
& \quad + \int_{\mathbb{R}^d} D_m U^0(t, x_0, m, y) \cdot D_p H(y, x_0, D_x U(t, y, x_0, m), m) dm(y) = 0 \\
& \qquad \text{in } (0, T) \times \mathbb{R}^{d_0} \times \mathcal{P}(\mathbb{R}^d), \\
(ii) & -\partial_t U - \Delta_x U - \Delta_{x_0} U + H(x, x_0, D_x U, m) - \int_{\mathbb{R}^d} \mathrm{div}_y D_m U(t, x, x_0, m, y) dm(y) \\
& \quad + D_{x_0} U \cdot D_p H^0(x_0, D_{x_0} U^0(t, x_0, m), m) \\
& \quad + \int_{\mathbb{R}^d} D_m U(t, x, x_0, m, y) \cdot D_p H(y, x_0, D_x U(t, y, x_0, m), m) dm(y) = 0 \\
& \qquad \text{in } (0, T) \times \mathbb{R}^d \times \mathbb{R}^{d_0} \times \mathcal{P}(\mathbb{R}^d), \\
(iii) & U^0(T, x_0, m) = G^0(x_0, m), \qquad \text{in } \mathbb{R}^{d_0} \times \mathcal{P}(\mathbb{R}^d), \\
(iv) & U(T, x, x_0, m) = G(x, x_0, m) \qquad \text{in } \mathbb{R}^d \times \mathbb{R}^{d_0} \times \mathcal{P}(\mathbb{R}^d).
\end{cases}
$$

where H^0 and H are the convex conjugate of L^0 and L : e.g.

$$
H^0(x_0, p_0, m) = \sup_{\alpha_0 \in \mathbb{R}^{d^0}} -\alpha_0 \cdot p_0 - L^0(x_0, \alpha^0, p_0)
$$

Lasry-Lions' results (2018)

- Existence/uniqueness in short time and discrete space for (**M**),
- Introduction of (**M**) with common noise,
- Scheme of proof for existence/uniqueness in short time for (**M**) (Hilbertian approach)

Alternative approach (C.- Cirant-Porretta, in preparation) :

- Existence/uniqueness for (**M**) in short time, **using a Trotter-Kato scheme**

Derivatives in the space of measures

We denote by $\mathcal{P}_2(\mathbb{R}^d)$ the set of Borel probability measures on $\mathbb{R}^d$ with finite second order moment, endowed for the Wasserstein distance

$$\mathbf{d}_2^2(m, m') = \inf_{\pi} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\pi(x, y),$$

where the infimum is taken over coupling between m and m'.

Derivatives

A map $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is C^1 if there exists a continuous and bounded map $\frac{\delta U}{\delta m} : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ such that, for any $m, m' \in \mathcal{P}(\mathbb{R}^d)$,

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}((1-s)m + sm', y) d(m' - m)(y) ds.$$

We set

$$D_m U(m, y) := D_y \frac{\delta U}{\delta m}(m, y).$$

Lemma

Let $U : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be smooth and $b : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d$ be a smooth and bounded (random) vector field. If m solves

$$dm_t = \{\Delta m_t + \mathrm{div}(b_t(x))\}dt,$$

then

$$\frac{d}{dt}U(m_t) = \int_{\mathbb{R}^d} \mathrm{div}(D_m U(m_t, y)m_t(dy) - \int_{\mathbb{R}^d} D_m U(m_t, y) \cdot b_t(y)m_t(dy).$$

Proof :

$$\begin{aligned}
\frac{d}{dt}U(m_t) &= \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m_t, y)\frac{d}{dt}m_t(y)dy \\
&= \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m_t, y)\{\Delta m_t(y) + \mathrm{div}(b_t(y))\}dy \\
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\end{aligned}$$

The verification result

Assumption : $H^0 = H^0(x_0, p_0, m)$ and $H(x, x_0, p, m)$ are strictly convex in p_0 and p respectively.

Proposition (C.-Cirant-Porretta)

Let (U^0, U) be a classical solution to the system of master equations (**M**). Then the pair

$$(\bar{\alpha}^0_t(x_0, m), \bar{\alpha}_t(x, x_0, m)) := -(D_pH^0(x_0, DU^0(t, x_0, m), m), D_pH(x, x_0, D_xU(t, x, x_0, m), m))$$

is a Nash equilibrium of the game (in the sense of Carmon and al.).

Proof

- Let α^0 be a competitor strategy for the major player and (X_t, m_t) solve

$$\left\{\begin{array}{l} dX_t^0 = \alpha_t^0(X_t^0, m_t)dt + \sqrt{2}dB_t^0 \\ d_t m_t = \left\{\Delta m_t + \mathrm{div}(m_t \bar{\alpha}_t(x, X_t^0, m_t))\right\} dt, \\ m_0 = \mu_0, \;\; X_0^0 = x_0^0. \end{array}\right.$$

  with $\bar{\alpha}_t(x, x_0, m) := -D_pH(x, x_0, D_xU(t, x, x_0, m), m)$.
- Then

$$\begin{aligned} dU^0(t, X_t^0, m_t) = & \Big\{\partial_t U^0 + D_{x_0}U^0 \cdot \bar{\alpha}^0 + \Delta_{x_0}U^0 + \int_{\mathbb{R}^d} \mathrm{div}_y(D_m U^0(t, X_t^0, m_t, y))m_t(dy) \\ & - \int_{\mathbb{R}^d} D_m U^0(t, X_t^0, m_t, y) \cdot D_pH(y, X_t^0, D_xU(t, y, X_t^0, m_t), m_t)m_t(dy)\Big\}dt \\ & + \sqrt{2}D_{x_0}U \cdot dB_t^0, \end{aligned}$$

  where U^0 is evaluated at (t, X_t^0, m_t).
- Recall that U^0 satisfies (**M**).

- Since U^0 satisfies (**M**), we have

$$\begin{aligned}\mathbf{E}\left[G^0(X^0_T,m_T)\right] &= \mathbf{E}\left[U^0(T,X^0_T,m_T)\right]\\ &= U(0,x_0,\mu_0)+\int_0^T \mathbf{E}\left[D_{x_0}U^0\cdot\alpha^0+H^0(X^0_t,D_{x_0}U^0(t,X^0_t,m_t),m_t)\right]dt\\ &\geq U(0,x_0,\mu_0)-\int_0^T \mathbf{E}\left[L^0(X^0_t,\alpha^0(t,X^0_t,m_t),m_t)\right]dt,\end{aligned}$$

- with an equality if

$$\alpha^0(t,X^0_t,m_t) = -D_pH^0(t,X^0_t,D_{x_0}U(t,X^0_t,m_t),m_t) = \bar{\alpha}^0(t,X^0_t,m_t),$$

  in which case $m=\bar{m}$.
- This shows that

$$\begin{aligned}J^0(\alpha^0;(m_t)) &= \mathbf{E}\left[\int_0^T L^0(X^0_t,\alpha^0_t(t,X^0_t,m_t),m_t)dt+G^0(X^0_T,m_T)\right]\\ &\geq U(0,x_0,\mu_0) = J^0(\bar{\alpha};(\bar{m}_t))\end{aligned}$$

  and proves the optimality of $\bar{\alpha}^0$.

Outline

The $N-$player game with a major player

- $N-$small players, 1 major player,
- Players play in feedback strategy : $\alpha^i = \alpha^i_t(x_0, \dots, x_N)$
- Goal of the players : to minimize
  - for the small players :

$$J^i(\alpha^0, \dots, \alpha^N) = \mathbf{E}\left[\int_0^T L(X^i_t, X^0_t, \alpha^i_t(\mathbf{X}_t), m^{N,i}_{\mathbf{X}_t})dt + G(X^i_T, X^0_T, m^{N,i}_{\mathbf{X}_T})\right],$$

  - for the major player :

$$J^0(\alpha^0, \dots, \alpha^N) = \mathbf{E}\left[\int_0^T L^0(X^0_t, \alpha^0_t(\mathbf{X}_t), m^N_{\mathbf{X}_t})dt + G^0(X^0_T, m^N_{\mathbf{X}_T})\right],$$

where $dX^i_t = \alpha^i_t(t, \mathbf{X}_t)dt + \sqrt{2}dB^i_t$, $\mathbf{X}_t = (X^0_t, \dots, X^N_t)$,
$m^{N,i}_{\mathbf{x}} = \frac{1}{N-1}\sum_{j\notin\{0,i\}} \delta_{x_j}$, $m^N_{\mathbf{x}} = \frac{1}{N}\sum_{j\neq 0}\delta_{x_j}$.

(Classical) verification

Let $(u^{N,0}, u^{N,1}, \dots, u^{N,N})$ solve the Nash system (**N**) :

$$\begin{cases} (i) & -\partial_t u^{N,0} - \displaystyle\sum_{j=0}^{N} \Delta_{x_j} u^{N,0} + H^0(x_0, D_{x_0} u^{N,0}, m_{\mathbf{x}}^N) \\ & \qquad + \displaystyle\sum_{j=1}^{N} D_{x_j} u^{N,0} \cdot D_p H(x_j, x_0, D_{x_j} u^{N,j}, m_{\mathbf{x}}^{N,j}) = 0, \\ (ii) & -\partial_t u^{N,i} - \displaystyle\sum_{j=0}^{N} \Delta_{x_j} u^{N,i} + H(x_i, x_0, D_{x_i} u^{N,i}, m_{\mathbf{x}}^{N,i}) \\ & +D_{x_0} u^{N,i} \cdot D_p H^0(x_0, D_{x_0} u^{N,0}, m_{\mathbf{x}}^N) + \displaystyle\sum_{j\neq i,\ j\geq 1} D_{x_j} u^{N,i} \cdot D_p H(x_j, x_0, D_{x_j} u^{N,j}, m_{\mathbf{x}}^{N,j}) = 0, \\ (iii) & u^{N,0}(T, \mathbf{x}) = G^0(x_0, m_{\mathbf{x}}^N),\ u^{N,i}(T, \mathbf{x}) = G(x_i, x_0, m_{\mathbf{x}}^{N,i}). \end{cases}$$

and let $\quad \alpha^0 = -D_p H^0(x_0, D_{x_0} u^{N,0}(t, \mathbf{x}), m_{\mathbf{x}}^N), \qquad \bar{\alpha}^i := -D_p H(x_i, x_0, Du^{N,i}(t, \mathbf{x}), m_{\mathbf{x}}^{N,i}).$

Then $(\bar{\alpha}^0, \bar{\alpha}^1, \dots, \bar{\alpha}^N)$ is a Nash equilibrium for the $N-$player game.

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Then $(\bar{\alpha}^0, \bar{\alpha}^1, \dots, \bar{\alpha}^N)$ is a Nash equilibrium for the $N-$player game.

The convergence result

Assumption : H^0 and H are globally Lipschitz continuous.

Theorem (C.-Cirant-Porretta)

Let $(u^{N,i})$ be a classical solution to the Nash system (**N**) and (U^0, U) be a classical solution to the system (**M**) of master equations (with bounded derivatives). There is a constant $C > 0$ such that

$$\left|u^{N,0}(t,\mathbf{x}) - U^0(t,x_0,m^N_{\mathbf{x}})\right| + \sup_{i=1,\dots,N}\left|u^{N,i}(t,\mathbf{x}) - U(t,x,x_0,m^{N,i}_{\mathbf{x}})\right| \leq CN^{-1}\left(1 + \frac{1}{N}\sum_{i=1}^{N}|x_i|\right),$$

where

$$m^N_{\mathbf{x}} = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}, \qquad m^{N,i}_{\mathbf{x}} = \frac{1}{N}\sum_{j\notin\{0,i\}}\delta_{x_j}.$$

The constant C is independent of N, $\mathbf{x} \in \mathbb{R}^{d_0} \times (\mathbb{R}^d)^N$ and $t \in [0,T]$.

Feature of proof

- Close to the ones in C.-Delarue-Lasry-Lions and Carmona-Delarue
- Classical difficulties :
  - no estimates,
  - relies on the smoothness of the solution (U^0, U) and on the uniformity of the noise,
- New difficulty : asymmetry of the players.

Conclusion

New results :

- A single notion of Nash equilibrium in closed loop form for MFG with a major player,
- New construction of the master equations
  (for MFG with common noise and for MFG with a major player),
- “Should be easily" extendable to
  - problems with several major players,
  - problems with a major player and a common noise.

Open questions :

- Structure (monotonicity ?) for long time existence of classical solution to (**M**),
- Notion of discontinuous solution for (**M**).

Thank you !

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