On the variational analysis for financial options with stochastic volatility

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Black and Scholes model: Given a financial asset $s(t)$, called the underlying, with dynamics

$$\mathrm{d}s = rs(t)\,\mathrm{d}t + \sigma s(t)\,\mathrm{d}W(t),$$

with volatility coefficient $\sigma \in \mathbb{R}$, interest rate $r \geq 0$, standard Brownian motion $W(t)$.

Value of European options are solutions of PDEs

$$\begin{cases} -u_t - rxu_x - \frac{1}{2}x^2\sigma^2 u_{xx} + ru = 0, & (x,t) \in \times(0,T) \times \mathbb{R}_+, \\ u(x,T) = u_T(x), & x \in \mathbb{R}_+ \end{cases}$$

with payoff u_T at final time T.

Payoff u_T given by call-option resp. put-option.

(i) Achdou-Tchou model

$$\begin{cases} \mathrm{d}s(t) = rs(t)\,\mathrm{d}t + \sigma(y(t))s(t)\,\mathrm{d}W_1(t), \\ \mathrm{d}y(t) = \theta(\mu - y(t))\,\mathrm{d}t + \nu\,\mathrm{d}W_2(t), \end{cases}$$

with interest rate r, volatility coefficient σ function of the factor y whose dynamics involve a parameter $\nu > 0$, and positive constants θ and μ.

(ii) Heston model

$$\begin{cases} \mathrm{d}s(t) & = & s(t)\left(r\,\mathrm{d}t + \sqrt{y(t)}\,\mathrm{d}W_1(t)\right), \\ \mathrm{d}y(t) & = & \theta(\mu - y(t))\,\mathrm{d}t + \nu\sqrt{y(t)}\,\mathrm{d}W_2(t). \end{cases}$$

- Y. Achdou and N. Tchou, ESAIM Math. Model. Numer. Anal., 2002.
- S. L. Heston, Rev. Financial Stud., 1993.

General model

$$\mathrm{d}X(t) = b(t, X(t))\,\mathrm{d}t + \sum_{i=1}^{n_\sigma} \sigma_i(t, X(t))\,\mathrm{d}W_i.$$

The associated elliptic operator

Second-order differential operator A corresponding to the general dynamics

$$Au := ru - b \cdot \nabla u - \tfrac{1}{2} \sum_{i,j=1}^{n_\sigma} \kappa_{ij} \sigma_j^\top u_{xx} \sigma_i,$$

Correlation coefficients $\kappa_{ij} : (0, T) \times \Omega \to \mathbb{R}$ between W_i and W_j, where for a partition (I, J) of $\{0, \ldots, N\}$ with $0 \in J$

$$\Omega := \prod_{k=0}^{N} \Omega_k; \quad \text{with} \quad \Omega_k := \begin{cases} \mathbb{R} & \text{when } k \in I, \\ (0, \infty) & \text{when } k \in J. \end{cases} \tag{1}$$

Associated backward PDE

Parabolic PDE

$$\begin{cases} -\dot{u}(t,x) + A(t,x)u(t,x) = f(x,t), & (t,x) \in (0,T) \times \Omega; \\ u(T,x) = u_T(x), & x \in \Omega, \end{cases}$$

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We want to apply the Lions–Magenes theory for well-posedness of the PDE.

Regularity results by Lions and Magenes

Given a Gelfand triple pair $V \subset H = H^* \subset V^*$, set

$$W(0,T) := \{u \in L^2(0,T;V) \ : \ \dot{u} \in L^2(0,T;V^*)\}.$$

Theorem (Lions-Magenes)

If $A(t) \in L^\infty(0,T;L(V,V^*))$ is uniform continuous and semi-coercive and $(f, u_T) \in L^2(0,T;V^*) \times H$, then:

- The PDE has a solution in $W(0,T)$ and the first parabolic estimate holds: for some $c > 0$ not depending on (f, u_T):

$$\|u\|_{L^2(0,T;V)} + \|u\|_{L^\infty(0,T;H)} \leq c(\|u_T\|_H + \|f\|_{L^2(0,T;V^*)}). \qquad (2)$$

- Under the additional hypothesis of semi-symmetry and $(f, u_T) \in L^2(0,T;H) \times V$ the second parabolic estimate holds: the solution $u \in W(0,T)$ belongs to $L^\infty(0,T;V)$, $\dot{u}$ belongs to $L^2(0,T;H)$, and for some $c > 0$ not depending on (f, u_T):

$$\|u\|_{L^\infty(0,T;V)} + \|\dot{u}\|_{L^2(0,T;H)} \leq c(\|u_T\|_V + \|f\|_{L^2(0,T;H)}). \qquad (3)$$

Semi-symmetry hypothesis

$$
\begin{cases}
A(t) = A_0(t) + A_1(t),\ A_0(t) \text{ and } A_1(t) \text{ continuous linear mappings } V \to V^*, \\
A_0(t) \text{ symmetric and continuously differentiable } V \to V^* \text{ w.r.t. } t, \\
A_1(t) \text{ is measurable with range in } H, \text{ and for positive numbers } \alpha_0,\ c_{A,1}: \\
\text{(i)} \quad \langle A_0(t)u, u\rangle_V \geq \alpha_0 \|u\|_V^2, \quad \text{for all } u \in V, \text{ and a.a. } t \in [0,T], \\
\text{(ii)} \quad \|A_1(t)u\|_H \leq c_{A,1}\|u\|_V, \quad \text{for all } u \in V, \text{ and a.a. } t \in [0,T],
\end{cases}
\tag{4}
$$

Bilinear form

Weighting function $\rho : \Omega \to \mathbb{R}_{>0}$ of class C^1, let

$$L^{2,\rho}(\Omega) := \{v \in L^0(\Omega);\ \int_\Omega v(x)^2 \rho(x)\,\mathrm{d}x < \infty\} \tag{5}$$

with norm

$$\|v\|_\rho := \left(\int_\Omega v(x)^2 \rho(x)\,\mathrm{d}x \right)^{1/2}. \tag{6}$$

$$Au := ru - b \cdot \nabla u - \tfrac{1}{2} \sum_{i,j=1}^{n_\sigma} \kappa_{ij} \sigma_j^\top u_{xx} \sigma_i, \tag{7}$$

where

$$\sigma_j^\top u_{xx} \sigma_i := \sum_{k,\ell=1}^{n_\sigma} \sigma_{kj} \frac{\partial u^2}{\partial x_k \partial x_\ell} \sigma_{\ell i}, \tag{8}$$

and for $v \in \mathcal{D}(\Omega)$:

$$a(u, v) := \int_\Omega Au(x) v(x) \rho(x)\,\mathrm{d}x \tag{9}$$

Lie derivative along a vector field

Let Φ be a vector field over Ω (i.e., a mapping $\Omega \to \mathbb{R}^n$).
The associated *Lie derivative* of $u : \Omega \to \mathbb{R}$ is

$$\Phi[u](x) := \sum_{i=0}^{n} \Phi_i(x) \frac{\partial u}{\partial x_i}(x), \quad \text{for all } x \in \Omega. \tag{10}$$

Bilinear form II

$$-\frac{1}{2}\int_{\Omega}\sigma_j^{\top}u_{xx}\sigma_i v\kappa_{ij}\rho = \sum_{p=0}^{3} a_{ij}^{p}(u,v), \tag{11}$$

Bilinear form II

$$-\tfrac{1}{2}\int_\Omega \sigma_j^\top u_{xx}\sigma_i v \kappa_{ij}\rho = \sum_{p=0}^{3} a_{ij}^p(u,v), \tag{11}$$

with

$$a_{ij}^0(u,v) := \tfrac{1}{2}\int_\Omega \sum_{k,\ell=1}^{n} \sigma_{kj}\sigma_{\ell i}\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_\ell}\kappa_{ij}\rho = \tfrac{1}{2}\int_\Omega \sigma_j[u]\sigma_i[v]\kappa_{ij}\rho, \tag{12}$$

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$$a_{ij}^1(u,v) := \tfrac{1}{2}\int_\Omega \sum_{k,\ell=1}^{n} \sigma_{kj}\sigma_{\ell i}\frac{\partial u}{\partial x_k}\frac{\partial(\kappa_{ij}\rho)}{\partial x_\ell}v = \tfrac{1}{2}\int_\Omega \sigma_j[u]\sigma_i[\kappa_{ij}\rho]\frac{v}{\rho}\rho, \tag{13}$$

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$$a_{ij}^2(u,v) := \tfrac{1}{2}\int_\Omega \sum_{k,\ell=1}^{n} \sigma_{kj}\frac{\partial(\sigma_{\ell i})}{\partial x_\ell}\frac{\partial u}{\partial x_k}v\kappa_{ij}\rho = \tfrac{1}{2}\int_\Omega \sigma_j[u](\operatorname{div}\sigma_i)v\kappa_{ij}\rho, \qquad (14)$$

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$$a_{ij}^3(u,v) := \tfrac{1}{2}\int_\Omega \sum_{k,\ell=1}^{n} \frac{\partial(\sigma_{kj})}{\partial x_\ell}\sigma_{\ell i}\frac{\partial u}{\partial x_k}v\kappa_{ij}\rho = \tfrac{1}{2}\int_\Omega \sum_{k=1}^{n}\sigma_i[\sigma_{kj}]\frac{\partial u}{\partial x_k}v\kappa_{ij}\rho. \tag{15}$$

Bilinear form

Contributions of the first and zero order terms resp. we get

$$a^4(u,v) := -\int_\Omega b[u]v\rho; \quad a^5(u,v) := \int_\Omega ruv\rho. \tag{16}$$

Set

$$a^p := \sum_{i,j=1}^{n_\sigma} a^p_{ij}, \quad p = 0,\dots,3. \tag{17}$$

The *bilinear* form associated with the above PDE is

$$a(u,v) := \sum_{p=0}^{5} a^p(u,v). \tag{18}$$

Semicoercivity of the principal term

For $\sigma = (\sigma_1, \dots, \sigma_{n_\sigma})$ the principal term of the bilinear form a is given by

$$a^0(u, v) = \sum_{i,j=1}^{n_\sigma} \int_\Omega \sigma_j[u]\sigma_i[v]\kappa_{ij}\rho = \int_\Omega \nabla u^\top \sigma\kappa\sigma^\top \nabla v \rho. \tag{19}$$

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Since $\kappa \succeq 0$, the above integrand is nonnegative when $u = v$; therefore, $a^0(u,u) \geq 0$. When $\kappa = \mathrm{id}$ we have that

$$a^{00}(u,u) := \int_\Omega |\sigma^\top \nabla u|^2 \rho = a^0(u,u). \tag{20}$$

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In the presence of correlations it is natural to assume that we have a coercivity of the same order. That is, we assume that

$$\text{For some } \gamma \in (0,1]: \quad \sigma\kappa\sigma^\top \succeq \gamma\sigma\sigma^\top, \quad \text{for all } (t,x) \in (0,T) \times \Omega. \tag{21}$$

Therefore, we have

$$a^0(u,u) \geq \gamma a^{00}(u,u). \tag{22}$$

Choice of Gelfand triple

Pair $V \subset H$ of Hilbert spaces, with dense inclusion.
For some measurable function $h : \Omega \to \mathbb{R}_+$ to be specified later we define

$$\left\{ \begin{array}{l} H := \{v \in L^0(\Omega); \;\; hv \in L^{2,\rho}(\Omega)\}, \\ \mathcal{V} := \{v \in H; \;\; \sigma_i[v] \in L^{2,\rho}(\Omega), \;\; i = 1, \ldots, n_\sigma\}, \\ V := \{\text{closure of } \mathcal{D}(\Omega) \text{ in } \mathcal{V}\}, \end{array} \right. \tag{23}$$

endowed with the natural norms,

$$\|v\|_H := \|hv\|_\rho; \quad \|u\|_V^2 := a^{00}(u,u) + \|u\|_H^2 . \tag{24}$$

Continuity of a^1

$$a^1_{ij}(u,v) = \tfrac{1}{2}\int_\Omega \sigma_j[u]\sigma_i[\kappa_{ij}\rho]\frac{v}{\rho}\rho, \tag{25}$$

and so,

$$\begin{aligned} |a^1_{ij}(u,v)| &\leq \sum_{j=1}^{n_\sigma} \|\sigma_j[u]\|_\rho \sum_{i=1}^{n_\sigma} \left\|\rho^{-1}\sigma_i[\kappa_{ij}\rho]v\right\|_\rho \\ &\leq C\|v\|_H \sum_{j=1}^{n_\sigma} \|\sigma_j[u]\|_\rho, \end{aligned} \tag{26}$$

whenever

$$\rho^{-1}\sum_i |\sigma_i[\kappa_{ij}\rho]| \leq C'h \tag{27}$$

It suffices that

$$\sum_i |\sigma_i[\kappa_{ij}]| + \rho^{-1}|\sigma_i[\rho]| \leq C''h. \tag{28}$$

Continuity of a^2

$$a^2_{ij}(u,v) = \frac{1}{2}\int_\Omega \sigma_j[u](\operatorname{div}\sigma_i) v\kappa_{ij}\rho, \tag{29}$$

and so,

$$\begin{aligned} |a^2_{ij}(u,v)| &\le \sum_{j=1}^{n_\sigma} \|\sigma_j[u]\|_\rho \sum_{i=1}^{n_\sigma} \|\operatorname{div}\sigma_i v\|_\rho \\ &\le C\|v\|_H \sum_{j=1}^{n_\sigma} \|\sigma_j[u]\|_\rho, \end{aligned} \tag{30}$$

whenever

$$\sum_i |\operatorname{div}\sigma_i| \le C'h. \tag{31}$$

Continuity of $a^{34} = a^3 + a^4$

$$a^{34}_{ij}(u,v) = \tfrac{1}{2} \int_\Omega \sum_{k=1}^{n} \sigma_i[\sigma_{kj}] \frac{\partial u}{\partial x_k} v \kappa_{ij} \rho - \int_\Omega b[u] v \rho = \int_\Omega q[u] v \rho, \tag{32}$$

At first sight, the continuity analysis needs a decomposition of the form

$$q = \sum_{k=1}^{n_\sigma} \eta_k \sigma_k \tag{33}$$

where η is of minimum Euclidean norm, and then we get

$$|a^{34}| \leq C \sum_{j=1}^{n_\sigma} \|\sigma_j[u]\|_\rho \|\eta_k v\|_{2,\rho} \leq \|v\|_H \sum_{j=1}^{n_\sigma} \|\sigma_j[u]\|_\rho, \tag{34}$$

whenever

$$|\eta| \leq Ch. \tag{35}$$

Continuity of a^5

Since

$$a^5(u, v) := \int_\Omega ruv\rho, \tag{36}$$

it is enough that

$$\sqrt{|r|} \leq Ch. \tag{37}$$

Synthesis for h

Factorizing the terms in a^1 and a^2 it suffices that for a.a. $x \in \Omega$:

$$\sum_i \left(|\sigma_i[\kappa_{ij}]| + \frac{|\sigma_i[\rho]|}{\rho} + |\operatorname{div} \sigma_i| \right) + |\eta| + \sqrt{|r|} \leq Ch. \tag{38}$$

Semi coercivity

By construction, for $i \in \{1, 2, 34, 5\}$

$$|a^i(u, v)| \leq C_i \|u\|_V \|v\|_H \tag{39}$$

and so,

$$\begin{aligned} a(u, u) &\geq \int_\Omega |\sigma^\top \nabla u|^2 \rho - C\|u\|_V \|u\|_H \\ &= \|u\|_V^2 - \|u\|_H^2 - C\|u\|_V \|u\|_H. \end{aligned} \tag{40}$$

The semicoercivity follows using Young's inequality.

Can we do better ?

Sometimes YES: using the notion of commutators of vector fields.

Commutators of vector fields

Let $u : \Omega \rightarrow \mathbb{R}$ be of class C^2. Let Φ and Ψ be two C^1 vector fields over Ω, both of class. Remember the *Lie derivative*

$$\Phi[u](x) := \sum_{i=0}^{n} \Phi_i(x) \frac{\partial u}{\partial x_i}(x), \quad \text{for all } x \in \Omega. \tag{41}$$

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Commutator of Φ and Ψ:

$$[\Phi, \Psi][u] := \Phi[\Psi[u]] - \Psi[\Phi[u]]. \tag{42}$$

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Note that

$$\Phi[\Psi[u]] = \sum_{i=1}^{n} \Phi_i \frac{\partial(\Psi u)}{\partial x_i} = \sum_{i=1}^{n} \Phi_i \left(\sum_{k=1}^{n} \frac{\partial \Psi_k}{\partial x_i} \frac{\partial u}{\partial x_k} + \Psi_k \frac{\partial^2 u}{\partial x_k \partial x_i} \right). \tag{43}$$

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So, the expression of the commutator is

$$[\Phi, \Psi]\,[u] = \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \Phi_i \frac{\partial \Psi_k}{\partial x_i} - \Psi_i \frac{\partial \Phi_k}{\partial x_i} \right) \frac{\partial u}{\partial x_k}. \tag{44}$$

This is the first-order differential operator associated with the Lie bracket of Φ, Ψ.

The adjoint to a vector field

Given two vector fields Φ and Ψ over Ω, define the spaces

$$\mathcal{V}(\Phi,\Psi) := \{v \in H;\ \ \Phi[v],\, \Psi[v] \in H\}\,, \tag{45}$$

$$V(\Phi,\Psi) := \{\text{closure of } \mathcal{D}(\Omega) \text{ in } \mathcal{V}(\Phi,\Psi)\}\,. \tag{46}$$

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We define the adjoint $\Phi^\top$ of Φ (viewed as an operator over say $C^\infty(\Omega, \mathbb{R})$), the latter being endowed with the scalar product of $L^{2,\rho}(\Omega)$), by

$$\langle \Phi^\top[u], v\rangle_\rho = \langle u, \Phi[v]\rangle_\rho \quad \text{for all } u, v \in \mathcal{D}(\Omega), \tag{47}$$

where $\langle \cdot, \cdot \rangle_\rho$ denotes the scalar product in $L^{2,\rho}(\Omega)$. Thus, there holds the identity

$$\int_\Omega \Phi^\top[u](x) v(x) \rho(x) \, \mathrm{d}x = \int_\Omega u(x) \Phi[v](x) \rho(x) \, \mathrm{d}x \quad \text{for all } u, v \in \mathcal{D}(\Omega). \tag{48}$$

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$$\int_\Omega \Phi^\top[u](x)v(x)\rho(x)\,\mathrm{d}x = \int_\Omega u(x)\Phi[v](x)\rho(x)\,\mathrm{d}x \quad \text{for all } u,v \in \mathcal{D}(\Omega). \tag{48}$$

Furthermore,

$$\begin{aligned}\int_\Omega u\sum_{i=1}^n \Phi_i \frac{\partial v}{\partial x_i}\rho\,\mathrm{d}x &= -\sum_{i=1}^n \int_\Omega v\frac{\partial}{\partial x_i}(u\rho\Phi_i)\,\mathrm{d}x \\ &= -\sum_{i=1}^n \int_\Omega v\left(\frac{\partial}{\partial x_i}(u\Phi_i) + \frac{u}{\rho}\Phi_i\frac{\partial \rho}{\partial x_i}\right)\rho\,\mathrm{d}x.\end{aligned} \tag{49}$$

The adjoint to a vector field II

Hence,

$$\Phi^\top[u] = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(u\Phi_i) - u\Phi_i \frac{\partial \rho}{\partial x_i}/\rho = -u \operatorname{div} \Phi - \Phi[u] - u\Phi[\rho]/\rho. \qquad (50)$$

We obtain that

$$\Phi[u] + \Phi^\top[u] + G_\rho(\Phi)u = 0, \qquad (51)$$

where

$$G_\rho(\Phi) := \operatorname{div} \Phi + \frac{\Phi[\rho]}{\rho}. \qquad (52)$$

Continuity of the bilinear form associated with the commutator

Setting, for v and w in $V(\Phi, \Psi)$:

$$\Delta(u, v) := \int_\Omega [\Phi, \Psi][u](x) v(x) \rho(x) \, \mathrm{d}x, \tag{53}$$

we have

$$\begin{aligned} \Delta(u, v) &= \int_\Omega (\Phi[\Psi[u]]v - \Psi[\Phi[u]]v)\rho \, \mathrm{d}x = \int_\Omega \Psi[u]\Phi^\top[v] - \Phi[u]\Psi^\top[v])\rho \, \mathrm{d}x \\ &= \int_\Omega \left(\Phi[u]\Psi[v] - \Psi[u]\Phi[v]\right) \rho \, \mathrm{d}x + \int_\Omega \left(\Phi[u]G_\rho(\Psi)v - \Psi[u]G_\rho(\Phi)v\right) \rho \, \mathrm{d}x. \end{aligned} \tag{54}$$

Lemma

For $\Delta(\cdot,\cdot)$ to be a continuous bilinear form on $V(\Phi,\Psi)$, it suffices that, for some $c_\Delta > 0$:

$$|G_\rho(\Phi)| + |G_\rho(\Psi)| \leq c_\Delta h \quad \textit{a.e.}, \tag{55}$$

and we have then:

$$|\Delta(u,v)| \leq \|\Psi[u]\|_\rho \left(\|\Phi[v]\|_\rho + c_\Delta \|v\|_H \right) + \|\Phi[u]\|_\rho \left(\|\Psi[v]\|_\rho + c_\Delta \|v\|_H \right). \tag{56}$$

Lemma

For $\Delta(\cdot,\cdot)$ to be a continuous bilinear form on $V(\Phi,\Psi)$, it suffices that, for some $c_\Delta > 0$:

$$|G_\rho(\Phi)| + |G_\rho(\Psi)| \leq c_\Delta h \quad \text{a.e.,} \tag{55}$$

and we have then:

$$|\Delta(u,v)| \leq \|\Psi[u]\|_\rho \left(\|\Phi[v]\|_\rho + c_\Delta \|v\|_H\right) + \|\Phi[u]\|_\rho \left(\|\Psi[v]\|_\rho + c_\Delta \|v\|_H\right). \tag{56}$$

We apply the previous results with $\Phi := \sigma_i$, $\Psi := \sigma_j$. Set for v, w in V:

$$\Delta_{ij}(u,v) := \int_\Omega [\sigma_i, \sigma_j][u](x)v(x)\rho(x)\,\mathrm{d}x, \quad i,j = 1,\ldots,n_\sigma. \tag{57}$$

We recall the definition $V = \{\text{closure of } \mathcal{D}(\Omega) \text{ in } \mathcal{V}\}$.

Corollary

Let (55) hold. Then the $\Delta_{ij}(u,v)$, $i,j = 1,\ldots,n_\sigma$, are continuous bilinear forms over V.

Redefining the space H

We now decompose q in the form

$$q = \sum_{k=1}^{n_\sigma} \eta_k'' \sigma_k + \sum_{1 \leq i < j \leq n_\sigma} \eta_{ij}' [\sigma_i, \sigma_j] \quad \text{a.e.} \tag{58}$$

We assume that η' and η'' are measurable functions over $[0, T] \times \Omega$, that η' is weakly differentiable, and that for some $c_\eta' > 0$:

$$h_\eta' \leq c_\eta' h, \text{ where } h_\eta' := |\eta''| + \sum_{i,j=1}^{N} \left| \sigma_i[\eta_{ij}'] \right| \quad \text{a.e.}, \;\; \eta' \in L^\infty(\Omega). \tag{59}$$

Lemma

Let (28), (31), (35), *and* (59) *hold. Then the bilinear form* $a(u,v)$ *defined in* (18) *is both* (i) *continuous and* (ii) *semi-coercive over* V.

Proof of (i).

(i) We only have to analyze the contribution of terms of the form setting $w := \eta'_{ij} v$ and taking here $(\Phi, \Psi) = (\sigma_i, \sigma_j)$, we get that

$$\int_\Omega \eta'_{ij}[\sigma_i, \sigma_j)[u] v \rho = \Delta(u, w), \tag{60}$$

where $\Delta(\cdot, \cdot)$ was defined in (53). Combining with lemma 1, we obtain

$$\begin{aligned} |\Delta_{ij}(u, w)| \leq \quad & \|\sigma_j[u]\|_\rho \left(\|\sigma_i[w]\|_\rho + c_\sigma \|\eta'_{ij}\|_\infty \|v\|_H \right) \\ & + \|\sigma_i[u]\|_\rho \left(\|\sigma_j[w]\|_\rho + c_\sigma \|\eta'_{ij}\|_\infty \|v\|_H \right). \end{aligned} \tag{61}$$

Since

$$\sigma_i[w] = \sigma_i[\eta'_{ij} v] = \eta'_{ij} \sigma_i[v] + \sigma_i[\eta'_{ij}] v, \tag{62}$$

by (59):

$$\|\sigma_i[w]\|_\rho \leq \|\eta'_{ij}\|_\infty \|\sigma_i[v]\|_\rho + \left\|\sigma_i[\eta'_{ij}] v\right\|_\rho \leq \|\eta'_{ij}\|_\infty \|\sigma_i[v]\|_\rho + c_\eta \|v\|_H. \tag{63}$$

Combining these inequalities, point (i) follows. □

Proof of (ii).

Use $u = v$ in (62) and (54). We find after cancellation in (54) that

$$\begin{aligned}\Delta_{ij}(u, \eta'_{ij}u) = & \int_\Omega u(\sigma_i[u]\sigma_j[\eta'_{ij}] - \sigma_j[u]\sigma_i(\eta'_{ij}))\rho \\ & + \int_\Omega (\sigma_i[u]G_\rho(\sigma_j) - \sigma_j[u]G_\rho(\sigma_i))\, \eta'_{ij}u\rho.\end{aligned} \tag{64}$$

By (59), an upper bound for the absolute value of the first integral is

$$\left(\|\sigma_i[u]\|_\rho + \|\sigma_j[u]\|_\rho \right) \|hu\|_\rho \leq 2 \|u\|_{\mathcal{V}} \|u\|_H . \tag{65}$$

By the same technique, we get $|\Delta_{ij}(u, \eta'_{ij}u)| \leq 4 \|u\|_{\mathcal{V}} \|u\|_H$. We finally have that for some $c > 0$

$$\begin{aligned} a(u,u) & \geq a_0(u,u) - c \|u\|_{\mathcal{V}} \|u\|_H , \\ & \geq a_0(u,u) - \tfrac{1}{2} \|u\|_{\mathcal{V}}^2 - \tfrac{1}{2}c^2 \|u\|_H^2 , \\ & = \tfrac{1}{2} \|u\|_{\mathcal{V}}^2 - \tfrac{1}{2}(c^2+1) \|u\|_H^2 . \end{aligned} \tag{66}$$

The conclusion follows. □

Remark: Similar statement in the case of the second parabolic estimate.

Application to stochastic volatility with multiple factor

$$
\begin{aligned}
\mathrm{d}s &= rs(t)\,\mathrm{d}t + \textstyle\sum_{k=1}^{N} |y_k(t)|^{\gamma_k} s^{\beta_k}(t)\,\mathrm{d}W_k(t), \\
\mathrm{d}y_k &= \theta_k(\mu_k - y_k(t))\,\mathrm{d}t + \nu_k |y_k(t)|^{1-\gamma_k}\,\mathrm{d}W_{N+k}(t), \quad k = 1, \dots, N.
\end{aligned} \tag{67}
$$

We assume that κ is constant and

$$
\beta_k \in (0,1]; \;\; \nu_k > 0; \;\; \gamma_k \in (0,\infty). \tag{68}
$$

Examples when $\beta_k = 1$: Heston $\gamma_k = \frac{1}{2}$, Tchou-Achdou $\gamma_k = 1$.
Assume that

$$
s\rho_s/\rho \in L^\infty; \;\; \rho_k/\rho \in L^\infty \text{ if } \Omega_k = \mathbb{R}; \;\; y_k\rho_k/\rho \in L^\infty \text{ if } \Omega_k = \mathbb{R}_+. \tag{69}
$$

Application to stochastic volatility with multiple factors

We get, assuming that $\gamma_1 \neq 0$, when all $y_k \in \mathbb{R}$, we can choose h' as

$$
\begin{aligned}
h' := \quad & 1 + \sum_{k=1}^{N} \left(|y_k|^{\gamma_k} (1 + s^{\beta_k - 1}) + (1-\gamma_k)|y_k|^{-\gamma_k} + |y_k|^{\gamma_k - 1} \right) \\
& + \sum_{k \in I} |y_k|^{1-\gamma_k} + \sum_{k \in J} |y_k|^{-\gamma_k}.
\end{aligned} \tag{70}
$$

Application to stochastic volatility with multiple factors

We get, assuming that $\gamma_1 \neq 0$, when all $y_k \in \mathbb{R}$, we can choose h' as

$$\begin{aligned} h' := \quad & 1 + \sum_{k=1}^{N} \left(|y_k|^{\gamma_k} (1 + s^{\beta_k - 1}) + (1 - \gamma_k) |y_k|^{-\gamma_k} + |y_k|^{\gamma_k - 1} \right) \\ & + \sum_{k \in I} |y_k|^{1-\gamma_k} + \sum_{k \in J} |y_k|^{-\gamma_k}. \end{aligned} \tag{70}$$

Without the commutator analysis we would get $h = h' + h''$, where

$$h'' := rs^{1-\beta_1}/|y_1|^{\gamma_1} + \sum_k \nu_k |\hat{\kappa}_k| |y_k|^{-\gamma_k}. \tag{71}$$

So, we have

$$h' \leq h, \tag{72}$$

meaning that it is advantageous to use the commutator analysis, due to the term $rs^{1-\beta_1}/|y_1|^{\gamma_1}$ in particular.
The second term has as contribution only for $\gamma_k \neq 1$ (since otherwise h' includes a term of the same order).

Heston case

For the generalized multiple factor Heston model (GMH), i.e. when $\gamma_k = 1/2$, $k = 1$ to N, we can take h equal to

$$h'_H := 1 + \sum_{k=1}^{N} \left(|y_k|^{\frac{1}{2}} (1 + s^{\beta_k - 1}) + |y_k|^{-\frac{1}{2}} \right), \tag{73}$$

when the commutator analysis is used, and when it is not, take h equal to

$$h_H := h_H + r s^{1-\beta_1} |y_1|^{-\frac{1}{2}}. \tag{74}$$

The original Heston model is for $k = 1$ and $\beta_1 = 1$.
So we get an improvement only when $\beta_k \neq 1$!

Weighting functions: Heston case

Lemma

(i) *For the GMH model, using the commutator analysis, in case of a* call option *with strike* K*, meaning that* $u_T(s) = (s-K)_+$*, we can take* $\rho = \rho_{call}$*, with*

$$\rho_{call}(s,y) := (1 + s^{\varepsilon''+3})^{-1} \Pi_{k=1}^{N} y_k^{\varepsilon'} (1 + y_k^{\varepsilon+2})^{-1}. \tag{75}$$

(ii) *For a* put option *with strike* $K > 0$*, we can take* $\rho = \rho_{put}$*, with*

$$\rho_{put}(s,y) := \Pi_{k=1}^{N} y_k^{\varepsilon'} (1 + y_k^{\varepsilon+2})^{-1}. \tag{76}$$

Perspectives

1. American option
2. Extension to other classes
3. Associated Fokker-Planck equations
4. Degenerate cases: Asian options

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