EXTENDED MEAN FIELD TYPE CONTROL AND APPLICATIONS

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Outline I

Outline II

- HJB EQUATION
- FINAL FORM

OBJECTIVES

1- FIRST MOTIVATION: Paper
Grid With Distributed Generation and Storage, C. Alasseur, I. Ben Tahar, A. Matoussi ,2017

2- SECOND MOTIVATION: Paper
Bellman Equation and Viscosity Solutions For Mean Field Stochastic Control, H. Pham, X. Wei,2015

3- MAIN OBJECTIVE:
Obtaining the Master Equation and the full theory of Extended Mean Field Type Control

DIFFICULTIES.

- Very complex nonlinear PDE in infinite dimension
- Complicated system of HJB-FP equations
- Examples can be tractable

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MEAN FIELD TYPE CONTROL I

Extends standard Stochastic Control Theory , by allowing dependence on the probability distribution of the state on the diffusion equation modeling the evolution of the state, and on the pay -off to be optimized. Control of McKean-Vlasov equations.
Extended Mean Field Type Control adds dependence on the probability distribution of the control on both the evolution of the state and the pay-off.

MAIN ELEMENTS OF THE THEORY I

In standard Stochastic Control Theory , Dynamic Programming leads to a single PDE, HJB equation
In Mean Field Type Control Theory, HJB equation is extended into a system of HJB-FP equations. In addition there is Bellman equation (not to be confused with HJB equation) and the most important one, the Master equation. From the Master equation, we can obtain the system of HJB-FP equations. The reverse is not true.
MAIN QUESTION: WHAT IS THE MASTER EQUATION IN EXTENDED MEAN FIELD TYPE CONTROL? WHAT IS THE SYSTEM OF HJB-FP EQUATIONS?

EXTENDED MEAN FIELD GAMES I

Mean Field Games differs from Mean Field Type control. It is not a stochastic control problem. It is a fixed point theory. There is no Bellman equation, but there is a Master equation and a system of HJB-FP equations. The interpretation is different. The starting point is the system of HJ-FP equations. Paradoxically, extension from Mean Field Games to Extended Mean Field Games is easier than extension from Mean Field Type Control to Extended Mean Field Type Control. Not discussed in this presentation.

Outline

BASIC DESCRIPTION I

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Following C. Alasseur et al. ,2017 in the simplest version, we consider a set of identical producers- consumers of energy. Each of them cannot influence the market, but we use the concept of representative agent, to aggregate the community. By considering that the representative agent is also the government, he has the power to impose his decisions. Naturally the problem of the representative agent must be very close to that of each individual agent, except for the fact that he takes into account the consequences of his decisions, which the individual agent cannot.
The first state variable $Q(t)$ of the representative agent is the net amount of power per unit of time that he produces. Net means production after substracting the consumption. This net rate of production is not controlled. What it means is that the production process is fixed and the consumption pattern is also fixed.

BASIC DESCRIPTION II

The evolution of $Q(t)$ is a diffusion

$$dQ = b(Q)dt + \sigma(Q)dw(t) \tag{1}$$

$$Q(0) = Q_0$$

where $w(t)$ is a standard Wiener process, and Q_0 is a random variable, independent of the Wiener process.

The control is in the possibility of storage . We call S the amount of energy which is stored. The control is a feedback $v(S,Q,t)$,abbreviated $v(S,Q)$, hence the evolution of $S(t)$

$$\frac{dS}{dt} = v(S,Q) \tag{2}$$

$$S(0) = 0$$

Therefore what is sent to the grid is in fact $Q(t) - v(S(t),Q(t),t)$ ar time t per unit of time.

Outline

PAYOFF FUNCTIONAL I

The average power sent to the grid is denoted by $E(Q(t)-v(S(t),Q(t)))$.The price paid by the grid is a decreasing function of this average . Following the paper of C. Alasseur et al we denote it $p(-E(Q(t)-v(S(t),Q(t))))$, assuming that $p(x)$ is a monotone increasing function. The random revenue from the storage strategy is thus $p(-E(Q(t)-v(S(t),Q(t))))(Q(t)-v(S(t),Q(t)))$. There are various costs to consider in front of this revenue. We choose to minimize, so the income enters in the payoff with a minus sign. We take the costs as in C. Alasseur et al.. We get the payoff functional to minimize

$$J(v(.))=E\int_0^T[\frac{a}{2}S(t)^2+lS(t)+\frac{c}{2}v^2(S(t),Q(t))+ \tag{3}$$

$$+\frac{K}{2}|Q(t)-v(S(t),Q(t))|^2]dt-$$

$$-\int_0^T p(-E(Q(t)-v(S(t),Q(t))))E(Q(t)-v(S(t),Q(t)))+Eh(S(T))$$

Outline

STATEMENT OF THE PROBLEM I

We have a state $x(t) = x_t \in R^n$. Its probability distribution is denoted by P_{x_t} . The control belongs to R^d and is defined by a feedback $v(x_t, P_{x_t})$. It also depends on time, but we omit the argument t, to simplify notation. The probability distribution of this control is denoted by $P_{v(x_t,P_{x_t})}$. The state evolution is defined by

$$dx = g(x_t, P_{x_t}, v(x_t, P_{x_t}), P_{v(x_t,P_{x_t})})dt + \sigma(x_t)dw(t) \tag{4}$$

$$x(0) = \xi$$

where ξ is a random variable, independent of the Wiener process $w(.)$

STATEMENT OF THE PROBLEM II

In (4) the drift is a function $g(x, m, v, \mu)$ where the arguments x,v are in R^n, R^d respectively and the arguments m, μ are probability measures on R^n, R^d respectively. Note that the probability $P_{v(x_t, P_{x_t})}$ is the image of P_{x_t} by the map $x \to v(x, P_{x_t})$. In the sequel, we use the notation $v(., m) * m$ for the image measure of m by the map $x \to v(x, m)$. So we write (4) as

$$dx = g(x_t, m_t, v(x_t, m_t), v(., m_t) * m_t)dt + \sigma(x_t)dw(t) \tag{5}$$

$$x(0) = \xi$$

with $m_t = m(t) = P_{x_t}$. We want to minimize the functional

$$J(v(.)) = E[\int_0^T f(x_t, m_t, v(x_t, m_t), v(., m_t) * m_t)dt + h(x_T, m_T)] \tag{6}$$

Outline

ANALYTIC FORMULATION I

We assume that ξ has a probability distribution, obtained from a density $m_0(x)$. We denote by $A = A(t)$ the second order differential operator

$$A\varphi(x) = -\mathrm{tr}a(x)D^2\varphi(x)$$

in which $a(x) = \frac{1}{2}\sigma(x)\sigma * (x)$. We call A^* the operator

$$A^*\varphi(x) = -\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(x)\varphi(x))$$

Under reasonable assumptions , the probability measure m_t will have a density $m(x,t)$, solution of the FP equation

$$\frac{\partial}{\partial t}m + A^*m + \mathrm{div}(g(x,m,v(x,m),v(.,m)*m)m(x,t)) = 0 \qquad (7)$$

$$m(x,0) = m_0(x)$$

ANALYTIC FORMULATION II

and the objective functional (6) can be written as follows

$$J(v(.)) = \int_0^T \int_{R^n} f(x, m_t, v(x, m_t), v(., m_t) * m_t) m(x, t) dxdt + \tag{8}$$

$$+ \int_{R^n} h(x, m_T) m(x, T) dx$$

where $m_t = m(x, t)$ is the solution of the FP equation (7). As usual we have reduced the original stochastic control problem to a deterministic control problem for a distributed parameter system, whose evolution is described by the FP equation (8).

Outline

FAMILY OF PROBLEMS I

To apply Dynamic Programming , we embed problem (7), (8) into the family

$$\frac{\partial}{\partial s}m+A^*m+\mathrm{div}(g(x,m,v(x,m),v(.,m)*m)m(x,s))=0,\ s>t \tag{9}$$

$$m(x,t)=m(x)$$

$$J_{m,t}(v(.))=\int_t^T\int_{R^n}f(x,m_s,v(x,m_s),v(.,m_s)*m_s)m(x,s)dxds+ \tag{10}$$

$$+\int_{R^n}h(x,m_T)m(x,T)dx$$

depending on the initial conditions m,t. We define the value function by

$$\Phi(m,t)=\inf_{v(.)}J_{m,t}(v(.)) \tag{11}$$

Outline

BELLMAN EQUATION I

The function $\Phi(m,t)$ is the solution of Bellman equation, provided derivatives can be defined. So our derivation is formal. The functional deriative $\partial_m\Phi(m,t)(x)$ is the function (omitting writing t) such that

$$\Phi(m') - \Phi(m) = \int_0^1 \partial_m\Phi(\theta m' + (1-\theta)m)(x)(m'(dx) - m(dx))d\theta \quad (12)$$

Bellman equation can be written as

$$-\frac{\partial\Phi}{\partial t} + \int_{R^n} A_x\partial_m\Phi(m,t)(x)m(x,t)dx = \quad (13)$$

$$= \inf_{v(.)}\int_{R^n}[f(x,m,v(x,m),v(.,m)*m)+$$

$$+D_x\partial_m\Phi(m,t)(x).g(x,m,v(x,m),v(.,m)*m)]m(x)dx$$

$$\Phi(m,T) = \int_{R^n} h(x,m)m(x)dx$$

BELLMAN EQUATION II

We shall use the notation

$$U(x,m,t) = \partial_m \Phi(m,t)(x) \tag{14}$$

So Bellman equation reads

$$-\frac{\partial \Phi}{\partial t} + \int_{R^n} A_x U(x,m,t)\, m(x,t) dx = \tag{15}$$

$$= \inf_{v(.)} \int_{R^n} [f(x,m,v(x,m),v(.,m)*m) +$$

$$+ D_x U(x,m,t).g(x,m,v(x,m),v(.,m)*m)]m(x)dx$$

and $\hat{v}(x,m) = \hat{v}(x,m,t)$ minimizes in $v(x,m)$ the functional

$$\int_{R^n} [f(x,m,v(x,m),v(.,m)*m) + \tag{16}$$

$$+ D_x U(x,m,t).g(x,m,v(x,m),v(.,m)*m)]m(x)dx$$

Outline

RULES OF DIFFERENTIATION I

Suppose we have a functional $\Psi(\mu)$ on probability measures on R^d and we consider $\Psi(v(.,m)*m)$, which we will consider first as a functional on $v(x,m)$ and then as a functional of m. We assume that Ψ has a functional derivative $\partial_\mu \Psi(\mu)(w)$, with $w \in R^d$, then we claim that

$$\frac{d}{d\theta}\Psi((v(.,m)+\theta\tilde{v}(.,m))*m)|_{\theta=0} = \tag{17}$$

$$\int_{R^n} D_w \partial_\mu \Psi(v(.,m)*m)(v(x,m)).\tilde{v}(x,m)m(x)dx$$

In the second rule , we consider the map $m \to \Psi(v(.,m)*m)$ and we look for its functional derivative. We claim the formula

$$\partial_m \Psi(v(.,m)*m)(x) = \partial_\mu \Psi(v(.,m)*m)(v(x,m)) + \tag{18}$$

$$+\int_{R^n} D_w \partial_\mu \Psi(v(.,m)*m)(v(\xi,m)).\partial_m v(\xi,m)(x)m(\xi)d\xi$$

EULER CONDITION I

Using the first rule of differentiation, we can write the Euler condition of optimality for the optimal feedback $\hat{v}(x,m)$ in the minimization of the functional (16). We first notice that , in (17)

$$\frac{d}{d\theta}\Psi((v(.,m)+\theta\tilde{v}(.,m))*m)|_{\theta=0}=0\,\forall\tilde{v}(.,m)\Longrightarrow$$

$$D_w\partial_\mu\Psi(v(.,m)*m)(v(x,m))=0$$

EULER CONDITION II

Therefore we get the relation

$$f_v(x,m,\hat{v}(x,m),\hat{v}(.,m)*m)+ \tag{19}$$

$$\int_{R^n} D_w\partial_\mu f(\xi,m,\hat{v}(\xi,m),\hat{v}(.,m)*m)(\hat{v}(x,m))m(\xi)d\xi+$$

$$+D_xU(x,m,t).g_v(x,m,\hat{v}(x,m),\hat{v}(.,m)*m)+$$

$$+\int_{R^n} D_\xi U(\xi,m,t)D_w\partial_\mu g(\xi,m,\hat{v}(\xi,m),\hat{v}(.,m)*m)(\hat{v}(x,m))m(\xi)d\xi$$

Outline

MASTER EQUATION I

The function $U(x, m, t)$ is the solution of

$$-\frac{\partial U}{\partial t} + A_x U(x, m, t) + \int_{R^n} A_\xi U(x, m, t)(x) m(\xi) d\xi = \tag{20}$$

$$f(x, m, \hat{v}(x, m), \hat{v}(., m) * m) + D_x U(x, m, t).g(x, m, \hat{v}(x, m), \hat{v}(., m) * m) +$$

$$+ \int_{R^n} [\partial_m f(\xi, m, \hat{v}(\xi, m), \hat{v}(., m) * m)(x) +$$

$$D_\xi U(\xi, m, t).\partial_m g(\xi, m, \hat{v}(\xi, m), \hat{v}(., m) * m)(x)] m(\xi) d\xi$$

$$+ \int_{R^n} D_\xi \partial_m U(\xi, m, t)(x).g(\xi, m, \hat{v}(\xi, m), \hat{v}(., m) * m) m(\xi) d\xi +$$

$$+ \int_{R^n} [\partial_\mu f(\xi, m, \hat{v}(\xi, m), \hat{v}(., m) * m)(\hat{v}(x, m)) +$$

$$+ D_\xi U(\xi, m, t).\partial_\mu g(\xi, m, \hat{v}(\xi, m), \hat{v}(., m) * m)(\hat{v}(x, m))] m(\xi) d\xi$$

MASTER EQUATION II

and the final condition

$$U(x,m,T) = h(x,m) + \int_{R^n} \partial_m h(\xi,m)(x) m(\xi) d\xi \tag{21}$$

and $\hat{v}(x,m)$ is related to $U(x,m,t)$ by the relation (19).

Outline

HJB-FP EQUATIONS I

With the feedback $\hat{v}(x,m,t)$ the FP equation becomes

$$\frac{\partial m}{\partial t} + A^* m + \mathrm{div}(g(x,m(t),\hat{v}(x,t),\hat{v}(.,t)*m(t))m(x,t)) = 0 \tag{22}$$

$$m(x,0) = m_0(x)$$

with $\hat{v}(x,t) = \hat{v}(x,m(t),t)$.We also define $u(x,t) = U(x,m(t),t)$. We can state that the function $\hat{v}(x,t)$ satisfies the Euler condition

$$f_v(x,m(t),\hat{v}(x,t),\hat{v}(.,t)*m(t)) + \tag{23}$$

$$+\int_{R^n} D_w \partial_\mu f(\xi,m(t),\hat{v}(\xi,t),\hat{v}(.,t)*m(t))(\hat{v}(x,t))m(\xi,t)d\xi +$$

$$+D_x u(x,t).g_v(x,m(t),\hat{v}(x,t),\hat{v}(.,t)*m(t)) +$$

$$+\int_{R^n} D_\xi u(\xi,t) D_w \partial_\mu g(\xi,m(t),\hat{v}(\xi,t),\hat{v}(.,t)*m(t))(\hat{v}(x,t))m(\xi,t)d\xi$$

HJB-FP EQUATIONS II

Finally , we obtain the HJB equation for $u(x,t)$

$$-\frac{\partial u}{\partial t}+Au=f(x,m(t),\hat{v}(x,t),\hat{v}(.,t)*m(t))+ \tag{24}$$

$$+D_x u(x,t).g(x,m(t),\hat{v}(x,t),\hat{v}(.,t)*m(t))$$

$$+\int_{R^n}[\partial_m f(\xi,m(t),\hat{v}(\xi,t),\hat{v}(.,t)*m(t))(x)+$$

$$+D_\xi u(\xi,t).\partial_m g(\xi,m(t),\hat{v}(\xi,t),\hat{v}(.,t)*m(t))(x)]m(\xi,t)d\xi+$$

$$+\int_{R^n}[\partial_\mu f(\xi,m(t),\hat{v}(\xi,t),\hat{v}(.,t)*m(t))(\hat{v}(x,t))+$$

$$+D_\xi u(\xi,t).\partial_\mu g(\xi,m(t),\hat{v}(\xi,t),\hat{v}(.,t)*m(t))(\hat{v}(x,t))]m(\xi,t)d\xi$$

HJB-FP EQUATIONS III

with the final condition

$$u(x,T) = h(x, m(T)) + \int_{R^n} \partial_m h(\xi, m(T)) m(\xi, T) d\xi$$

Outline

INTERPRETATION I

We interpret all functions

$$g(S,Q,m,v,\mu) = \begin{vmatrix} v \\ b(Q) \end{vmatrix}, \ \sigma(x) = \begin{vmatrix} 0 \\ \sigma(Q) \end{vmatrix} \tag{25}$$

$$A\varphi(S,Q) = -\frac{1}{2}\sigma^2(Q)\frac{\partial^2\varphi}{\partial Q^2} \tag{26}$$

$$f(S,Q,m,v,\mu) = \frac{a}{2}S^2 + lS + \frac{c}{2}v^2 + \frac{K}{2}|Q-v|^2 - \tag{27}$$

$$-p(-\int qm(s,q)dsdq + \int w\mu(dw))(Q-v)$$

$$h(x,m) = h(S) \tag{28}$$

INTERPRETATION II

We obtain the derivatives

$$f_v(S,Q,m,v,\mu) = (c+K)v - KQ + p(-\int qm(s,q)dsdq + \int w\mu(dw)) \tag{29}$$

$$\partial_m f((S,Q,m,v,\mu))(s,q) = p'(-\int qm(s,q)dsdq + \int w\mu(dw))q(Q-v)$$

$$\partial_\mu f((S,Q,m,v,\mu))(w) = -p'(-\int qm(s,q)dsdq + \int w\mu(dw))w(Q-v)$$

and

$$g_v(S,Q,m,v,\mu) = \left| \begin{array}{l} 1 \\ 0 \end{array} \right. \tag{30}$$

$$\partial_m g((S,Q,m,v,\mu))(s,q) = 0,\ \partial_\mu g((S,Q,m,v,\mu)) = 0$$

EQUATION OF THE MEAN I

We use the notation

$$\rho(t) = \int \hat{v}(s, q, t) m(s, q, t) ds dq \tag{31}$$

$$\bar{Q}(t) = \int q m(s, q, t) ds dq = \int q m(q, t) dq$$

where $m(q, t)$ is the probability density of the diffusion (1) . We can then note that

$$f_v(S, Q, m(t), \hat{v}(S, Q, t), \hat{v}(, t) * m(t)) = \tag{32}$$

$$(c + K)\hat{v}(S, Q) - KQ + p(\rho(t) - \bar{Q}(t))$$

$$\partial_\mu f((S, Q, m(t), \hat{v}(S, Q, t), \hat{v}(, t) * m(t))(w) = \tag{33}$$

$$+p'(\rho(t) - \bar{Q}(t)) w(\hat{v}(S, Q) - Q)$$

EQUATION OF THE MEAN II

$$
\begin{aligned}
&\partial_m f((S,Q,m(t),\hat{v}(S,Q,t),\hat{v}(,t)*m(t))(s,q) = \\
&-p'(\rho(t)-\bar{Q}(t))q(\hat{v}(S,Q)-Q) \\
&\int D_w \partial_\mu f((s,q,m(t),\hat{v}(s,q,t),\hat{v}(,t)*m(t))(\hat{v}(S,Q))m(s,q,t)dsdq = \\
&p'(\rho(t)-\bar{Q}(t))(\rho(t)-\bar{Q}(t))
\end{aligned}
\tag{34}
$$

EQUATION OF THE MEAN III

Introduce the function

$$\zeta(x) = p(x) + xp'(x) \tag{35}$$

then the Euler condition (23) becomes

$$(c+K)\hat{v}(S,Q) - KQ + \frac{\partial u}{\partial S} + \varsigma(\rho(t) - \bar{Q}(t)) = 0 \tag{36}$$

Multiplying by $m(S,Q,t)$ and integrating we get also

$$(c+K)\rho(t) - K\bar{Q}(t) + \int \frac{\partial u}{\partial S}(s,q,t)m(s,q,t)dsdq + \varsigma(\rho(t) - \bar{Q}(t)) = 0 \tag{37}$$

Next

$$f(S,Q,m(t),\hat{v}(S,Q,t),\hat{v}(,t)*m(t)) = \frac{a}{2}S^2 + lS + \frac{c}{2}\hat{v}(S,Q)^2 + \tag{38}$$

$$+\frac{K}{2}|Q - \hat{v}(S,Q)|^2 + p(\rho(t) - \bar{Q}(t))(\hat{v}(S,Q) - Q)$$

EQUATION OF THE MEAN IV

$$\int \partial_m f((s,q,m(t),\hat{v}(s,q,t),\hat{v}(,t)*m(t))(S,Q)dsdq = \tag{39}$$

$$-p'(\rho(t)-\bar{Q}(t))Q(\rho(t)-\bar{Q}(t))$$

$$\int \partial_\mu f((s,q,m(t),\hat{v}(s,q,t),\hat{v}(,t)*m(t))(\hat{v}(S,Q))dsdq = \tag{40}$$

$$+p'(\rho(t)-\bar{Q}(t))\hat{v}(S,Q)(\rho(t)-\bar{Q}(t))$$

Outline

REWRITING HJB EQUATION I

The HJB equation (24) becomes

$$-\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2(Q)\frac{\partial^2 u}{\partial Q^2} - \frac{\partial u}{\partial S}\hat{v}(S,Q) - \frac{\partial u}{\partial Q}b(Q) = \frac{a}{2}S^2 + lS + \frac{c}{2}\hat{v}(S,Q)^2 + \tag{41}$$

$$+\frac{K}{2}|Q - \hat{v}(S,Q)|^2 + (\hat{v}(S,Q) - Q)\zeta(\rho(t) - \bar{Q}(t))$$

$$u(S,Q,T) = h(S)$$

and the FP equation is

$$\frac{\partial m}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial Q^2}(\sigma^2(Q)m) + \frac{\partial}{\partial S}(\hat{v}(S,Q)m) + \frac{\partial}{\partial Q}(b(Q)m) \tag{42}$$

$$m(S,Q,0) = \delta(S)\bigotimes m_0(Q)$$

where $m_0(Q)$ is the probability distributtion of the initial value Q_0.

Outline

HJB AND GRADIENT EQUATION I

Combining , we simplify HJB equation

$$-\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2(Q)\frac{\partial^2 u}{\partial Q^2} - Q\frac{\partial u}{\partial S} - b(Q)\frac{\partial u}{\partial Q} + \tag{43}$$

$$+\frac{c+K}{2}|\hat{v}(S,Q) - Q|^2 = \frac{a}{2}S^2 + \frac{c}{2}Q^2 + lS$$

$$u(S,Q,T) = h(S)$$

We then define $\lambda(S,Q,t) = \dfrac{\partial u}{\partial S}(S,Q,t)$. Differentiating (43) with respect to S and combining we obtain

$$-\frac{\partial \lambda}{\partial t} - \frac{1}{2}\sigma^2(Q)\frac{\partial^2 \lambda}{\partial Q^2} - \hat{v}(S,Q)\frac{\partial \lambda}{\partial S} - b(Q)\frac{\partial \lambda}{\partial Q} = aS + l \tag{44}$$

$$\lambda(S,Q,T) = h'(S)$$

From (44) and (42) , we can infer

$$\int \lambda m(s,q,t)dsdq = \int h'(s)m(s,q,T)dsdq + l(T-t) + \tag{45}$$

$$+a\int_t^T \left(\int sm(s,q,\tau)dsdq\right)d\tau$$

On the other hand, from the FP equation we get also

$$\int sm(s,q,t)dsdq = \int_0^t \rho(\tau)d\tau \tag{46}$$

After rearrangements

$$(c+K)\rho(t) - K\bar{Q}(t) + \varsigma(\rho(t) - \bar{Q}(t)) + \tag{47}$$

$$+\int h'(s)m(s,q,T)dsdq + l(T-t) + a\int_0^T (T-t)\vee\tau\rho(\tau)d\tau = 0$$

In this equation , the probability $m(s,q,T)$ stills intervenes and the full system HJB-FP and (47) remains coupled.

We get a decoupling when the function $h(S)$ is quadratic. We then assume

$$h(S) = h_0 \frac{S^2}{2} + h_1 S + h_2 \tag{48}$$

and we get from (47) an integral equation for the function $\rho(t)$, namely

$$(c+K)\rho(t) + \varsigma(\rho(t) - \bar{Q}(t)) + a\int_0^T (T-t) \vee \tau \rho(\tau) d\tau \tag{49}$$

$$+h_0 \int_0^T \rho(\tau) d\tau = K\bar{Q}(t) - l(T-t) - h_1$$

Knowing the function $\rho(t)$ we can obtain the function $\lambda(S, Q, t)$ by solving (44) , taking account of (36).

$$-\frac{\partial \lambda}{\partial t} - \frac{1}{2}\sigma^2(Q)\frac{\partial^2 \lambda}{\partial Q^2} - b(Q)\frac{\partial \lambda}{\partial Q} + \frac{1}{c+K}\lambda\frac{\partial \lambda}{\partial S} - \tag{50}$$

$$-\frac{1}{c+K}(KQ - \varsigma(\rho(t) - \bar{Q}(t)))\frac{\partial \lambda}{\partial S} = aS + l$$

$$\lambda(S, Q, T) = h_0 S + h_1$$

We can then get the feedback

$$\hat{v}(S,Q,t) = \frac{KQ - \lambda(S,Q,t) - \varsigma(\rho(t) - \bar{Q}(t))}{c+K} \tag{51}$$

When $\hat{v}(S,Q,t)$ is known, then the function $u(S,Q,t)$ is solution of (43) which is a linear problem. We can obtain $\lambda(S,Q,t)$ as

$$\lambda(S,Q,t) = \lambda_0(t)S + \lambda_1(Q,t) \tag{52}$$